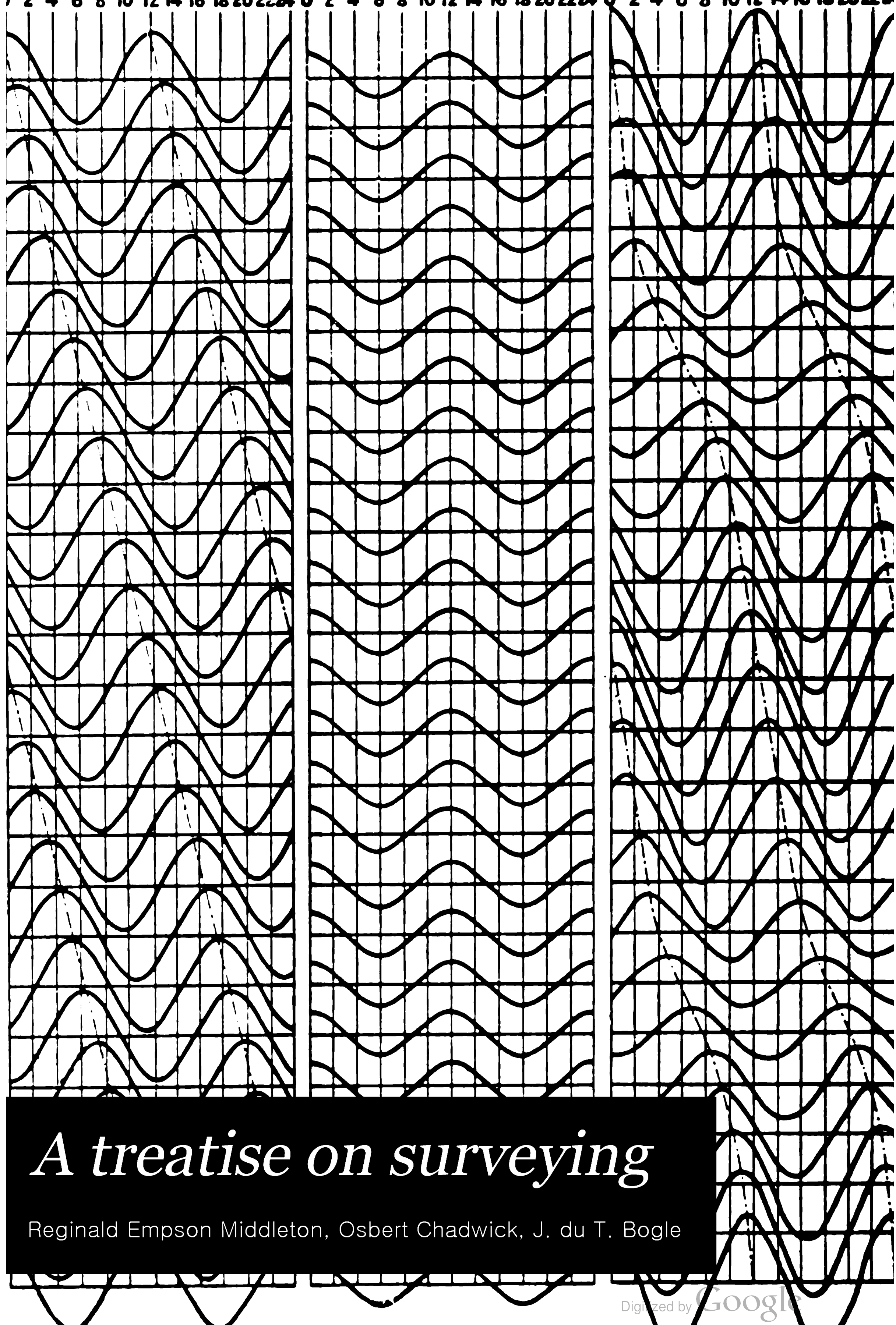

A treatise on surveying

Reginald Empson Middleton, Osbert Chadwick, J. du T. Bogle

FIG. 115

A TREATISE ON SURVEYING

A TREATISE ON

SURVEYING

COMPILED BY

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CONTENTS

OF

THE SECOND PART.

CHAPTER I.

DRAWING INSTRUMENTS AND DRAWING MATERIALS.

CHAPTER II.

DESCRIPTION AND ADJUSTMENT OF INSTRUMENTS.

(CONTINUED FROM CHAP. III. PART I.)

CHAPTER III.

TACHEOMETRIC SURVEYING.

CHAPTER IV.

BASE LINE MEASUREMENTS, AND METHODS OF CHAINMENT WHERE GREAT ACCURACY IS REQUIRED.

CHAPTER V.

RAILWAY CURVES.

CHAPTER VI.

SPHERICAL TRIGONOMETRY.

CHAPTER VII.

THE FIGURE OF THE EARTH.

CHAPTER VIII.

PRELIMINARY INVESTIGATIONS AND PROBLEMS IN PHYSICAL AND PRACTICAL ASTRONOMY.

CHAPTER IX.

DETERMINATION OF LATITUDE, TIME AND AZIMUTH.

CHAPTER X.

ROUTE SURVEYS AND RECONNAISSANCE.

CHAPTER XI.

PLANE-TABLE TOPOGRAPHY.

CHAPTER XII.

SKETCHING, OR RAPID SURVEY WORK.

CHAPTER XIII.

BAROMETRIC AND HYPSOMETRIC DETERMINATIONS OF HEIGHTS ABOVE SEA-LEVEL.

CHAPTER XIV.

PROJECTIONS AND SUNDIALS.

CHAPTER XV.

GAUGING STREAMS AND RIVERS.

CHAPTER XVI.

SURVEYS OF COAST LINES, HARBOURS, ROADSTEADS, RIVERS, ETC.

CHAPTER XVII.

TIDAL PHENOMENA.

APPENDICES :

PREFACE

TO

THE SECOND PART.

THE circumstances which led to the compilation of the Work, of which this is the Second (and final) Part, have been fully set forth in the First Part, already published.

The First Part describes the instruments which are usually in the possession of the surveyor, and gives the methods of using them in surveying with the chain only, with the chain and theodolite on traverse surveying, and in minor triangulation. The determination of heights by levelling and by the measurement of vertical angles, is also dealt with. Only a few brief remarks have been made as to the effect of the spherical form of the earth on survey operations. The Second Part, now published, describes some modern and special instruments not so generally possessed by surveyors, such as the tacheometer.

The question of the Figure of the Earth is discussed more fully mainly with the intention of indicating the limits within which the spheroidal form of the earth may or may not be disregarded. The object of this section is mainly suggestive. No attempt has been made to give full information as to the methods to be adopted in the case of surveys of so extensive a character as to involve the consideration of the spheroidal figure of the earth. The occasions on which a surveyor is called upon to conduct such extensive surveys are rare, and their direction is usually entrusted to specially qualified persons.

A section on Practical Astronomy has been inserted, with a view of instructing the surveyor as to the performance of the astronomical operations which are demanded of him, namely, the determination of latitude and azimuth, which can be effected by means of the instruments which he normally possesses. In the preliminary remarks and

definitions at the commencement of this section the *apparent* motions of the heavenly bodies have alone been discussed. To have considered their real physical motions would have unduly extended this treatise; and, after all, it is the apparent motions only that have to be observed in the solution of problems of practical astronomy.

As the solution of astronomical problems involves the use of Spherical Trigonometry, a short section on this subject has been added, in which the leading and most useful formulæ are given.

The present volume also contains a section, mainly written by the late General WOODTHORPE, C.B. (appointed Surveyor-General of India, just before his death), describing the special methods employed in the conduct of Exploratory Surveys, as used by himself and others, in Topographical Surveys of the northern frontiers of India.

To render the work useful to the Civil Engineer, a section has been added on the measurement of the Flow of Streams and Rivers, an operation which has often to be performed in connection with surveys, especially those for projects of water supply.

For similar reasons a section on Curve-ranging has been introduced.

In connection with the section on Marine Surveying, some remarks are made as to the general character of tidal phenomena. The writers make no apology for not going into this abstruse question more fully. Their object in discussing tidal phenomena is merely to inform the surveyor as to the general character of the phenomena which he may expect to observe, and as to the observations which he should make, in order to determine mean sea-level, and to afford to the specialist the data for predicting the times of high and low water respectively. To have explained the operation of deducing tidal constants from observed data would have been beyond the scope of this work. Even were the surveyor to perform this operose work, the results would be of no appreciable use to him, for he could not practically predict times and heights of high and low water without the use of a tide-predicting machine, of which but two or three exist. The proper course for the surveyor to pursue, therefore, is to place the observations in the hands of an expert, one who has not only facilities for reducing the elements but who also has the use of a tide-predicting machine.*

The main reason which induced the writers to approach the subject of tidal motion is the fact that, in many parts of the world, the simple

* Mr. F. Roberts, of the Nautical Office, to whom the writers are indebted for much of the information as to tidal phenomena, undertakes the reduction of tidal observations and the construction of tide tables for the Indian and other Governments.

rules for prediction given in works on navigation, which practically suffice in British waters, are utterly useless, and yet, when harmonic analysis has been applied to adequate data, correct prediction has proved to be possible.

When a work, like the present one, has been compiled by several writers in the leisure moments of active professional life, it is scarcely possible to avoid occasional repetitions. Nor can a work, produced as this has been, by several collaborators, be as concise, consecutive and symmetrical as if it had been wholly written by one. The writers, therefore, crave indulgence for any defects in these respects that may be found by the reader.

In conclusion, the writers would record their obligation to Colonel BOGLE, late R.E., for his assistance in editing and for the articles which he has contributed, mainly that on Practical Astronomy. They also desire to express their obligation to Mr. T. ORMSBY for his most valuable assistance both in editing and in the correction of the proofs, a most laborious task in the present case, owing to the great number of mathematical formulæ and calculations.

PREFACE TO THE THIRD EDITION.

THIS edition has been revised and several clerical errors have been eliminated. Chapter V. on Railway Curves has been rewritten and enlarged. Four new Appendices have been added, including one dealing with photography from kites and balloons, and one giving descriptions of some new instruments which have been put on the market since the last edition was published. Several of the figures have been improved.

SURVEYING.

PART II.

CHAPTER I.

DRAWING INSTRUMENTS & DRAWING MATERIALS.

In this chapter, a short description is given of the drawing instruments, in the use of which the student may not have been already instructed. No description will therefore be given of the following, as they are fully described in either 'Drawing Instruments,' by Stanley, or 'Mathematical Instruments,' by Heather, viz.—

Drawing or ruling pens, dividers or compasses, protractor or parallel ruler, all of which are found in any good case of instruments.

Drawing Board. A good drawing board is a great convenience, though by no means so necessary in a surveyor's equipment, as it is in that of the architectural draughtsman, or engineer.

A good board is made up of narrow slips of very well seasoned white pine, clamped at the back in such a manner as to allow of the slips being closed together as shrinkage takes place. The surface should be true, and the edges perfectly straight, and exactly at right angles to each other. A slip of ebony is sometimes let into one or both of the ends, on which the Tee-square travels more evenly than on the cross grain of the wood—2 feet 4 inches by 3 feet 6 inches, is a convenient size.

Tee-Square. The Tee-square, which can only be used on a drawing board, is useful for ruling horizontal or vertical lines, or for guiding an offset scale when plotting sections, etc. The square should be of the same length as the board with which it is used, the stock being about one-third of its length. Pear-wood or mahogany, edged with ebony, make good Tee-squares, but ebony alone soils the paper.

Drawing Paper. The drawing paper used for finished survey sheets, should be of the best quality obtainable, and should never be wetted, stretched, or mounted on a drawing board.* Above all things, unequal expansion or contraction must be guarded against, and to this end the paper should be kept lying flat in a drawer in loose sheets, when possible,

* Although surveys should never be plotted on paper, mounted, by gluing to a drawing-board, on account of the distortion that takes place when the stretched paper is cut off, one of the writers has found it advantageous to mount the paper as for a water-colour drawing. When dry, the sheet is cut off and put away flat in a drawer. This takes out the buckles of the paper, and renders it pleasant to work upon. *Vide* Part I., page 144.

where it will be affected to a minimum extent by changes in the hygrometric conditions of the atmosphere. When used, it should be pinned to the drawing board or table with drawing pins.

The most useful sizes of drawing paper are designated as follows.

Imperial—30 inches by 22 inches.

Double elephant—40 inches by 27 inches.

Antiquarian—53 inches by 31 inches.

Cartridge paper is sometimes useful, but it does not take colours or tints very well. Continuous cartridge is obtainable either 53 or 60 inches wide and of any length up to 300 yards.

Straight-edge. For plotting survey work over an extended area, and for ruling lines of latitude and longitude, a steel Straight-edge is very necessary. Wooden straight-edges are not recommended for this purpose. Clamps are sometimes useful for fixing a straight-edge in one position on a drawing board when plotting a section or offsets from a traverse line. One edge of the flat steel should be bevelled.

Set-Squares. It is well to have a few Set-squares of various sizes, if not relying on a set of Marquois scales, hereafter described. The former consist of triangular pieces of thin wood, vulcanite, or celluloid, one angle being always 90°, *i.e.* a right angle, the other complementary angles are varied, though those most frequently adopted are each 45°, or 30° and 60° respectively As to material, a transparent celluloid is found to be most satisfactory.

A set-square is used in conjunction with a straight-edge or tee-square, as a base.

Marquois' Scales. These scales seem to be very little used nowadays by draughtsmen, being regarded as chiefly useful for military drawings. For drawings of a limited size, a set of Marquois scales supplies the place of the set-square, the straight-edge, and the parallel-ruler. The set consists of two scales of equal width, each a litttle over 1 foot in length (leaving about ⅛th of an inch beyond each terminal graduation), and a set-square or triangle, the whole being made in stout box-wood of about ⅛th of an inch in thickness.

The triangle has two of its sides of a length in the proportion of 3 to 1, the longest being the hypothenuse and the shortest the base. The remaining side is bevelled for use with the drawing pen. There is an index in the centre of the longest side, which reads into the scales on the rulers.

On each of the edges of the pair of rulers, a pair of scales is inscribed, the inner one being divided into many parts of an inch, from 20 to 60. The divisions of each outer scale are three times the length of the divisions of the corresponding inner scale. The former are for use with the triangle, whose hypothenuse and base are in the ratio of 3 to 1, so that if the index above mentioned be moved along the divisions of the outer scale, the bevelled edge can be used for ruling lines at intervals apart, corresponding to the divisions on the inner or natural scale.

Scales. Draughtsmen's scales are usually made about a foot long, of either box-wood or ivory—other materials such as metal, vulcanite,* celluloid, or paper, are sometimes used. The latter are supposed to expand and

* Both ivory and vulcanite expand and contract largely, the former with moisture, the latter with heat.

contract with the drawing paper, but to this end it is best to construct a scale for each survey, and on each sheet of paper, as is done at the foot of each sheet of the Ordnance Survey maps.*

The section of scales, which are usually divided on the bevelled edge, are either flat, oval, or triangular, an improved American pattern of the latter section consisting in hollowing out the centre portions and fining down the solid angles, clearness in the divisions being ensured by the use of a white enamel ground.

Protractors, Military Pattern.

Every surveyor must possess some means of protracting angles measured, if only for making a rough sketch of the area or route surveyed.

The ordinary military protractor is 6 inches long, and 3 inches wide, the divisions to 1° being numbered twice, viz. the outer figures with the hands of a clock from 10° to 170° and the inner figures in the same direction from 190° to 350°. On no account should the inner figures be affixed in the inverse order to the outer ones. A scale is engraved on the remaining edge. The body of this protractor is filled with scales and horizontal equivalents for use when contouring, and transverse parallel lines are ruled for setting it in azimuth when plotting. If a small hole be bored near the scale edge, and a weight be suspended by a thread, this instrument can be used as a rough clinometer. This protractor is usually made of box-wood, though the narrower patterns are still made of ivory. To get the greatest length per degree, on an average, it is obvious that a rectangular protractor should have a width of half its length.

Circular, Semicircular, or Quadrant Protractors.

Protractors are made of various radii to three patterns, viz. circular, semicircular, or quadrant, divided on the outer bevelled edges to 360°, 180°, and 90°, respectively. For general use the semicircular protractor is to be preferred. A circular protractor made of transparent parchment paper is useful for office work.

Protractor with Vernier and Arm.

When it is desired to plot angles to one minute of arc, it is necessary to use a vernier and arm. The arm is folding and carries a sharp pointer. The protractor and arm are usually made of brass, the scale and vernier being silvered. Four short sharp studs on the underside, prevent the protractor from slipping when once carefully set in position. The centre mark is usually made of glass, on which two lines, at right angles to one another, are engraved, and inked up, their intersection being the exact centre. Instead of the centre being defined by the intersection of two lines, a very small circular hole may be cut out, leaving the mark on the plan over which it is desired to place the centre, clear and well defined, the eye judging its true centering. This protractor is sometimes provided with two arms, and two verniers, as well as a clamp and tangent screw, for accurate setting.

The Station Pointer.

The station pointer is practically a metal, circular protractor, with one fixed, and two movable arms, which latter are fitted with clamps, slow-motion screws, and verniers, to provide for accurate setting to any desired angle. This instrument is used for plotting the observer's position when angles between three fixed points have been observed with a sextant,

* *Vide* remarks on plotting, Part I., page 144.

† A protractor, with graduations, numbered in opposite directions, is sometimes convenient when working with the sextant, for example when plotting soundings, by the three-point method, with a sheet of tracing-paper.

or other angle-measuring instrument. In fact this instrument solves graphically the three-point problem. The method of using the station pointer, and setting the arms, is too obvious to require describing.

This instrument is peculiarly adapted for plotting the positions of soundings taken when making a marine survey.

Instruments for ruling Parallel Lines. Besides the tee-square, which can only be used with a drawing board, there are several appliances for ruling parallel lines on a plan.

Plain Parallel Ruler. The plain parallel ruler requires no special description, except that to ensure its working correctly, it is necessary to be sure that the distances between the pivots of each of the two bars be *exactly* equal, as well as the distances between the same pivots on each slide rule.

Double-barred Parallel Ruler. In this pattern, three instead of two rules are connected, by which arrangement the ruling edge can be moved over a greater distance from the fixed edge, and it moves moreover in a direct line from the same. This system is not susceptible of great accuracy, as the errors inherent in the plain parallel ruler are here augmented.

Rolling Parallel Ruler. On a good drawing board, the rolling parallel ruler is capable of producing very accurate work. The rule runs on two grooved wheels of the same diameter and connected by a metal rod, which raise it very slightly from the paper. The axle is protected by a metal bridge which covers it. The best rulers of this description are made of gun-metal or electrum, and as the ruler is raised from the paper, the latter is not soiled by the metal.*

The accuracy of this instrument can be readily tested by reversing it between two runs of ten inches or so, and noting the divergence, if any, of the two lines drawn at the end of each run.

Beam Compass. The beam compass is the most accurate instrument for either setting off distances over a few inches in length, or for striking arcs of from 15 to 60 inches in length. The most simple form consists in a bar of well-seasoned mahogany to which two beam heads (as they are termed) are fitted. One of these heads is fixed to one end of the bar, and is provided with a clamp and slow-motion screw for making fine adjustments, as well as a tube and screw for holding one of the pointers, or a pen or pencil point as required. The other head which also carries a pointer, pen or pencil, can be moved to any part of the bar, and then clamped.

The scale sometimes provided, should not be used except for trial or rough measurements, since distances laid off by its use cannot be relied on. All distances should be taken direct from a scale drawn on the paper on which the plan is being plotted, or else some other reliable standard of length.

A beam compass should always be used to test the perpendicularity of the central meridian line of a survey to a parallel of latitude, and to test the rectangular margins of a survey sheet, by measuring distances of 3 and 4 (or their

* The rolling parallel-rulers, as supplied by most makers, are not heavy enough. The bridge over the axle of the rollers should be displaced by a semi-cylindrical mass of metal, say, $1\frac{1}{4}$ inch diameter; or, two bars should be soldered on to give sufficient weight. The bright electrum also is painful to the eyes, and therefore the ruler should be given a dead bronze.

multiples) on the perpendiculars, and testing the hypothenuse with the distance 5 (or its multiple).

The length of a standard yard, if one is handy, can be multiplied along a pavement or suitable even surface till a distance of one chain has been laid off, for use in conducting the daily tests of the chain's length, before and after work.

The Pantagraph.

The pantagraph is an instrument chiefly used for the reduction of plans, though slight enlargements can also be drawn, as well as copies to the same scale. All errors are magnified when used for enlargements, rendering its use for this purpose undesirable.

This instrument is built up of four arms or rulers of stout brass, one pair being half the length of the other. The longer pair are jointed at one end, and the shorter pair, which are also jointed together at one end, have their other ends jointed to the centres of the long pair of rulers. A castor is placed under each joint to support it, and to allow of the whole frame being freely moved and extended over the surface of the paper.

A carrier holding a tracing point is fixed near the free end of one of the long rulers, for use when reductions are being made.

The free end of the other long ruler carries a sliding head similar to that fitted to a beam compass, and is provided with a screw clamp to fix it at the required position.

A similar sliding head is fitted to the adjoining short arm, and both of these heads have sockets to fit either a fulcrum pin, a tracing point, or a pen or pencil. A heavy weight with a vertical brass pin is used as a pivot or fulcrum, and this can be used with either of the last named heads, and round it the whole instrument then moves. The pen or pencil carrier can be weighted, and can be raised by means of a silk cord, when desired. The bars fitted with the sliding heads, are each graduated and marked $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., indicating the reduction effected when both heads are set to the same fraction. When the instrument is correctly set, the fulcrum, tracing-point, and pencil, will be in one straight line.

To copy a plan to the same scale, the fulcrum must be placed in the central position, and when enlarging the tracing-point must take this position.

The fractions $\frac{1}{2}$, $\frac{1}{3}$, etc. above referred to, produce reductions in the proportions stated, but not so with the figures $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, etc., which reduce in the ratio of the *numerator* to the *sum of the numerator and denominator* or as 2 to 5, 3 to 7, 4 to 9, etc.

The Eidograph.

The eidograph performs the same operations as the pantagraph, and possesses several advantages over the latter instrument. It has but one, in the place of 4 or 6 supports on the drawing, it is more regular in its actions, and the joints being fulcrums fitted in accurately ground bearings, the regularity and accuracy of the motion round them are susceptible of more accurate adjustment. The eidograph can also be used to reduce plans in any desired proportion whatever, instead of following fixed fractions.

The point of support is a heavy weight, having a few points on the under side to prevent it from slipping on the paper, whilst on the upper side a vertical fulcrum is fixed, round which the instrument revolves. A socket fitting the fulcrum is attached to a sliding head, which has a clamping screw to fix it any desired position, as it moves along the centre beam of the instrument. Two wheels

of the same diameter revolve on pivots fixed at the ends of the beam, and these revolve similarly, being attached by steel bands working on their circumferences. An arm sliding in a socket is attached to the under side of each of the wheels.

A pencil or tracer is carried at the extremity of each rod. A sliding loose weight is provided to adjust the balance of the instrument when required. The beam, and both the arms, are graduated from the centre outwards from 0 to 100, which can be read to $\frac{1}{1000}$ by means of the verniers cut in the boxes.

The rules for setting the instrument are usually pasted to the lid of its case.

The following answers for ordinary proportions. 'Multiply the *difference* between the denominator and numerator of a given proportion by 100, and divide the multiplicand or result by the *sum*.'

For example. To reduce to $\frac{3}{4}$ of the original,

$$\frac{100\,(4-3)}{4+3} = \frac{100}{7} = 14 \cdot 28 \text{ nearly,}$$

and this number must be set on all the three arms.

Again. To reduce to $\frac{3}{5}$ of the original,

$$\frac{100\,(5-3)}{5+3} = \frac{200}{8} = 25,$$

which must be set on each arm.*

The Computing Scale. The computing scale is the instrument almost universally employed, on account of its simplicity and reliability, for computing the areas of plots of land delineated on a large scale survey. For use with plans to small scales, this instrument is rather clumsy, and in such cases the planimeter (described later on) is to be preferred.

The computor consists of a ruler of boxwood about 20 inches long, with a groove cut down its centre to act as a guide to the slide.

The slide is a light metal frame with a small handle fixed to it for manipulation. Across the centre of this frame, and perpendicular to the ruler, a fine platinum wire or horse-hair is fixed.

The scale on the top of the ruler, into which a mark on the carrier reads, is so divided as to read acres and roods when used with lines ruled at a given fraction of an inch apart (usually $\frac{1}{3}$rd).

Computors are constructed and scaled to suit plans drawn to various scales. The following is the mode of using one. The portion of the plan whose area has to be computed, is either covered with parallel lines ruled in pencil at the intervals for which the instrument is constructed, or else a piece of transparent tracing paper similarly ruled with black or red lines is placed over it. The computor is now placed parallel to and with its upper edge close to the top line, which cuts off a part of the area in question, and with its index set to zero on the scale. If an odd-shaped piece is formed by the boundary, the wire of the carrier must be placed so as to include and exclude the same amount, as judged by the eye. The slide or carrier is then traversed to the left, and stopped on that

* The graduations on the limbs, both of the pantagraph and eidograph, must not be trusted implicitly. It should be ascertained by experiment that a line of given length is reduced to another of lesser length in the desired proportion.

boundary in a similar manner as when starting. This process is continued in space after space between consecutive lines, care being taken not to move the index when the computor is shifted from left to right. The summation of the readings gives the area (*vide also* Part I.).

Amsler's Planimeter. For the computation of small areas, especially if bounded by irregular outlines, this little instrument is all that can be desired as regards simplicity of construction, rapidity in computing, and accuracy.

This instrument consists of two arms jointed together with a wheel attached to one of the arms, by the motion of which the area circumscribed by the pointer attached to the extremity of the same arm is recorded. A pointer attached to the end of the other arm forms a centre, round which the instrument revolves.

It is not proposed to enter into the details of the construction or use of the instrument, which are set forth in the works on drawing instruments above quoted. Those who wish to follow the subject theoretically are referred to a paper by Mr. F. P. Purvis in the 'Philosophical Magazine' for July, 1874, or to that by Sir F. M. Bramwell, C.E., in the 'British Association Reports' for 1872, page 401.

The accuracy of the instrument can be tested by passing the 'tracer' over the circumference of a known area or a circle to a given radius.

There are two kinds of planimeters in general use.

(1) The *fixed* planimeter, which works to one scale only, reading in square inches, and is most useful for measuring the area of indicator diagrams.

(2) The *proportional* planimeter, in which the unit can be changed by altering the length of the arm which carries the 'tracer.'*

The Opisometer. The opisometer is an instrument for measuring the lengths of winding roads, paths, etc., on a map, and consists of a small mill-edged wheel, revolving on a screw, which forms its axle. The axle is carried by guides fixed to an ivory or bone handle. The use of this little instrument is obvious, but it may be noted that it requires to be run back to zero over the scale of the plan to get the distance measured.

Chartometer or Wealemefna. This little measuring instrument, which is a little over the size of a shilling, is used for running off measurements on a plan, giving the result in feet and inches. It consists of a clockwork mechanism, with an indicating arm, actuated by a small wheel which rolls on the paper, and measures length with passable accuracy. Made in gold or silver it may form a pendant to a watch-chain.

Brushes. Satisfactory shading or colouring can only be done with sable-hair brushes. The hairs should be of a light reddish colour, long and stiff. A good brush when wetted should form a well defined conical point. There are about seven sizes of these brushes mounted in quills termed crow, duck, goose, swan and eagle, the prices varying from 4*d.* to 13*s.*

* The graduations on the arm of the proportional planimeter cannot be relied upon implicitly. In the first place it will often happen that when set to the desired scale of area the tracing-point will not pass conveniently round the area to be measured. The best plan is to draw, very carefully, preferably on the plan itself, a square or rectangle, to the actual scale of the plan, representing some round number of area-units. Then, setting the arm to a convenient length, pass round the figure several times, and thus determine the coefficient for the particular setting.

CHAPTER II.

DESCRIPTION AND ADJUSTMENT OF INSTRUMENTS.

(CONTINUED FROM CHAPTER III., PART I.)

Hadley's 8-inch Sextant.

THIS instrument is similar in principles of construction to the 'box sextant' described in Chapter III., Part I. It is well adapted to the purposes of the explorer, being portable and easy of manipulation, though, from being held in the hand, the extreme accuracy of fixed instruments is not to be expected of it.*

The 8-inch sextant, reading to 10″ of arc, is probably the best size, though the 6-inch, also reading to 10″, is preferred by some on account of increased portability.

Principle of the Sextant.

The angle between the two objects whose images are made to coincide in the field of view of the telescope, is measured in the plane containing them and the observer, and is twice the angle obtained between the 'horizon' and 'index' glasses. This latter fact is evident from the construction of the instrument, for though the index-arm may be pushed forward only 60° of arc, the extreme angle *read* on the limb is 120°, all the angles on the limb being doubled. The proof of the above statement is here given:—

Let the ray from an object S, (S M fig. 1), be reflected at M (from the surface of the index mirror) to the horizon glass at H, where it is again reflected to the eye-piece or telescope of the instrument. Here it coincides with the direct ray O H, as seen through the un-silvered half of the horizon glass.

From the laws of reflection we know that the angle S M P is equal to the angle P M H. Let these angles be called x. Similarly the angles M H Q and E H Q are equal. Let these be called y.

Now the exterior angle S M H is equal to the sum of the angles M H E and H E M, or

$$\angle HEM = 2x - 2y$$

Similarly

$$\angle HQM = x - y$$

$$\therefore \quad \angle HEM = 2 \angle HQM$$

But the angle H Q M is the angle between the normals to the mirrors, and is therefore equal to the angle H Q′ M, the angle between the mirrors.

* Stands are provided for use with sextants. These consist of a tribrach with an upright pillar, on the top of which is a sort of universal joint, carrying a transverse arm, to one end of which the sextant is screwed, the opposite end carrying a weight, balancing the sextant. With this stand, great steadiness may be obtained. The weight of the sextant and stand, however, is not much less than that of a theodolite, on the whole the more convenient instrument.

The following are the adjustments of the 8-inch Hadley's sextant :—

Adjustments of the 8-inch Sextant.

1. Index-glass, for perpendicularity.
2. Horizon-glass, for perpendicularity.
3. Parallelism of sight-line of telescope to plane of sextant.
4. Index-correction.

1. *Index-glass for Perpendicularity.*—This adjustment is effected when the silvered surface of the glass is perpendicular to the plane of the sextant. Set the index-arm near the middle of the arc, then, placing the eye near the index-glass, observe whether the arc seen directly, and its reflected image in the glass, appear

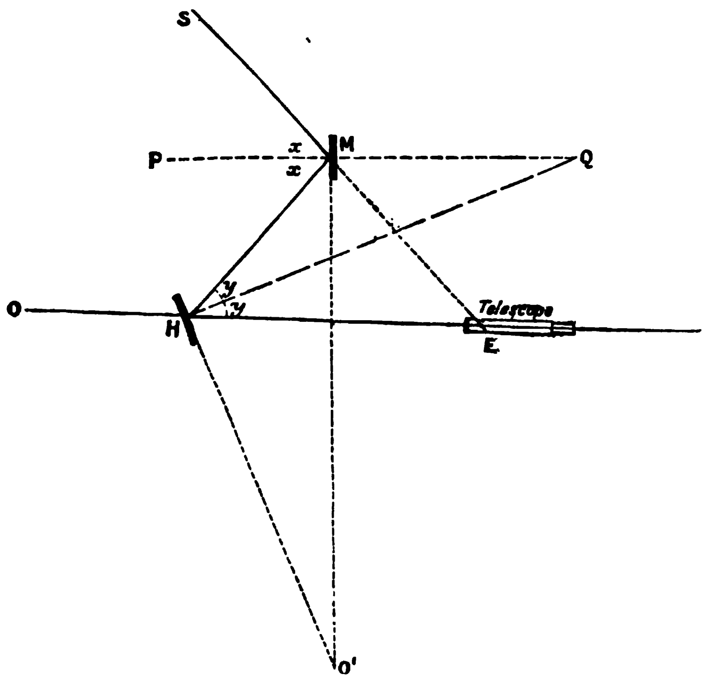


FIG. 1.

to form one continuous arc, which will be the case *only* when this adjustment is perfect. The index-glass can be set up by loosening the small screws which fix the plate that carries the glass, and wedging it forwards, or backwards, with a piece of paper, as found necessary.

2. *Horizon-glass for Perpendicularity.*—To effect this adjustment, the same conditions must be fulfilled as with the index-glass. Having completed the 'first' adjustment, direct the telescope to a star, and move the index-arm until the reflected image of the star appears to pass the direct image. If one image passes on either *side* of the other, the horizon-glass is not parallel to the index-glass, and must be adjusted by means of the small screw provided for that purpose, until one image passes over the other. For the above adjustment, a star of the third magnitude will afford greater precision than the brighter ones. Any distant well-defined terrestrial object may be substituted for the star, though the latter is to be preferred when the sextant is required for astronomical observations.

Many observers prefer to adjust their sextants so that one image passes just clear of the other.

3. *Parallelism of Sight-line of Telescope to Plane of Sextant.*—Turn the sliding tube of the telescope round till two of the wires are parallel to the plane of the sextant. Select two stars, or the moon and a star not less than 90° apart, and bring them into con..ct on the right-hand wire. Then keeping the index clamped, change the position of the sextant till they appear on the other wire. If the contact remain perfect the line of sight midway between the wires is parallel to the plane of the instrument, and *vice versâ*. The adjustment of the telescope is effected by means of two small opposing screws in the ring which carries it.

If the objects appear to separate on the wire *further* from the index-arm, the object-glass end of the telescope droops towards it, and *vice versâ*.

Observations for Index Correction.

4. *Index-correction.*—As the index-correction for a sextant is continually varying with the temperature, it is necessary to determine it both *before* and *after* each set of observations. The method of procedure is as follows.

(*a*) *With the Sun.*—Clamp the index at about 30 minutes from zero on the arc, look towards the sun, and perfect the contact of the two images by means of the tangent screw. Note the reading of the vernier (say 31′ 56″). Then clamp the index to about 30 minutes from zero on the arc of excess, and as before, perfect the contact of the two images. Note the reading of the vernier (say 31′ 22″). Half the difference of these two readings is the index-correction, *additive* when the reading *off* the arc (*i.e.*, on the arc of excess) is the greater of the two, and *vice versâ*.

Example.

Reading on arc	=	31′ 56″
„ off arc	=	31′ 22″
Difference	=	34″
Index-correction *subtractive*	=	17″

This observation may be checked by comparing *one-fourth of the sum of the two readings* with the sun's semi-diameter for the day given in page ii. of the month in the N.A. In the above example the sun's semi-diameter by observation would be 15′ 49″·5.

(*b*) *With a Star.*—Bring the direct and reflected images of the star into coincidence, and read the arc. This reading is the index-correction, *additive* if *off* the arc (*i.e.*, on the arc of excess), and *subtractive* if *on* the arc.

Errors of Eccentricity and Graduation.

Besides index-error, sextants are subject to errors of eccentricity and graduation. The first arises from the centres of the arc and the index-arm not being truly identical. The observer may eliminate its effects by a judicious pairing of observations, *i.e.*, when observing meridional stars for latitude, choose pairs of stars at about equal distances north and south of the zenith, and similarly, when observing ex-meridional stars for time or azimuth, choose pairs at about equal altitudes east and west of the meridian, adopting in each case the mean of a pair of results for a single determination.

The *graduation* of a sextant may be measured by comparing the coincidence

of the ends of the vernier with the graduation of the scale at different parts of the arc. This will not, however, detect errors of centering.

The sextant is *par excellence* an instrument suited to nautical requirements, and is of the greatest value in making maritime surveys of coasts, harbours, estuaries, etc., and wherever soundings have to be taken, and mapped.

Altitudes of Heavenly Bodies.

The altitudes of heavenly bodies are measured with the sextant at sea, from the 'sensible horizon,' but on land, it is necessary to use an 'artificial horizon,' the plane of whose surface coincides with that of the 'sensible horizon.'

Artificial Horizon.

The angle observed with the 'artificial horizon,' is however double the altitude of the body observed, as is evident from an inspection of fig. 2. The mercurial artificial horizon is undoubtedly the best form. It consists of a tray 5 inches by $2\frac{5}{8}$ inches internal dimensions, into which the mercury is poured from a metal bottle. It is covered by

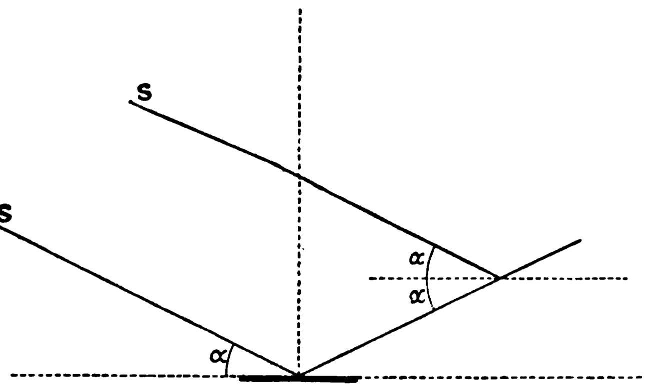


FIG. 2.

a glass roof-like shade, to protect the surface from agitation in windy weather. This shade must be made of plate glass, with truly parallel face, or the rays passing through them will be unequally refracted.. Test this by taking some observations, first *with*, and then *without* the cover. The course of the rays from the *upper* and *lower* limbs of the sun, when the artificial horizon is used, is shown in fig. 3.*

Instructions for Observing Altitudes of Heavenly Bodies with the Sextant.

In order to ensure accuracy it is undesirable to trust to a single observation, but a set of observations must be taken, following one another *as closely as possible*, and the mean of these is used for calculation. The time taken therefore in completing a set of observations is a factor of importance in astronomical observing, and considerable practice is necessary before a proper standard of accuracy and rapidity is attained.

* The great merit of the sextant as an instrument for measuring celestial altitudes is the absolute certainty of the mercurial horizon. This can only be affected by want of parallelism in the plate-glass of the cover. This error may be eliminated to some extent by reversing. The undulation of the mercury-surface, owing to the slight earth-tremor, is often annoying. One of the writers has found it necessary to go to a distance of a mile or more from a camp, in order to be clear of the tremors caused by the stamping of the horses.

The following instructions for observing with the sextant should be followed :—

1. Focus the telescope before you put it in the socket.

2. Place the wires of the telescope parallel and perpendicular to the plane of the instrument.

3. When observing the angle between two objects of different brightness, always, if possible, direct the telescope to the fainter.

4. When observing the altitude of the sun, use the dark-glass on the eye-piece in preference to the movable dark-glasses, in order to avoid errors arising from the possible non-parallelism of the surfaces of the latter.

5. When observing the altitude of the sun, take the mean of alternate observations (or sets of observations) of the upper and lower limb, in preference to applying the correction for semi-diameter.

The reason of this is that different observers do not take 'contacts' in precisely the same way, some taking it at moments when the contact is more

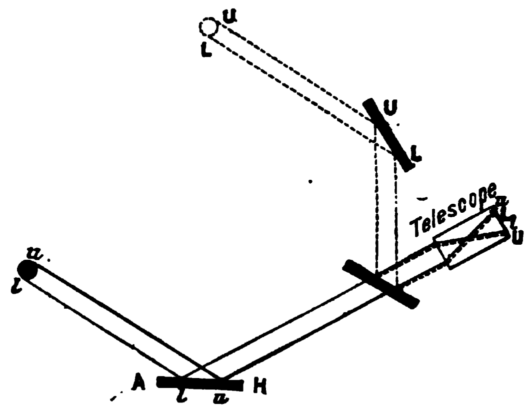


FIG. 3.

fully developed than others. If, however, the same observer takes *both* upper and lower limbs in the same manner, the mean of the two observations will be correct, whereas the application of the semi-diameter value to one observation might not give him so accurate a result.

6. When the altitude of the celestial body is changing rapidly, determine roughly the arc through which the body moves in the time required to complete an observation. Set the vernier at successive divisions at this interval apart, and record the times at which the body makes contact. This will give altitudes at equal intervals of arc. When observing in the meridian or near the pole, this method is impracticable.

7. It is essential for good observing that the images be defined by careful focussing, that they should be so faint as not to fatigue the eye, and that their intensities should be as nearly as possible equal.

8. Care must be taken in observing the sun, to be certain which limb is being observed. The limb which precedes the other as regards motion in the sky does so in the field of the telescope also.

9. To observe well with the sextant it is necessary to accustom oneself to

place it quickly in the true plane, and this can only be done by practice. When observing the sun, by casting the eye down to the concave surface of the graduated arc, it can be seen at once if the instrument is being correctly held, as if so, the shadow of the lower bar will be parallel to the edge of the arc.

10. To 'bring down' a heavenly body, arrange yourself in front of the artificial horizon in such a position that when the sextant is brought down to the position for observing, your eye will be properly situated for seeing the image through the telescope in the artificial horizon. Set the sextant (without clamping) to zero, and direct the telescope towards the heavenly body, so that you can see both images in the field of view, then, holding the index arm in the left hand, keep the reflected image always in the field of view, whilst you turn down the telescope with the right hand till it points to the artificial horizon, when the direct image will be readily found there.

11. Observations for index-correction should be taken *before* and *after* using the sextant for taking an observation.

12. When using the artificial horizon—

(*a*) Reverse the cover between each set of observations, to eliminate errors arising from possible non-parallelism of the surfaces of the glass.

(*b*) The mercury should be carefully strained, and its surface kept free from dirt. A little loose earth or sand scraped together and resting on a firm substratum forms a good bed for the artificial horizon, but care must be taken that in reversing the cover the sand is not allowed to foul the surface of the mercury.

(*c*) Both the *inner* and *outer* surfaces of the glass must be wiped occasionally and kept free from moisture, which often forms gradually whilst the observations are in progress, especially at night time.

Identity of Stars.

13. In observing stars with the artificial horizon it is sometimes difficult for the unpractised observer to be sure that he sees the same star through the horizon glass and the index-glass. *Certainty* of identity can *only* be obtained by 'bringing down' the star in the usual way. It must be remembered in this respect that the 'direct vision' star is reversed in azimuth but *not* in altitude, the 'reflected' star being reversed in *both* azimuth and altitude.

Sextant-stand.

Observations with the sextant may be rendered more accurate by the use of a *sextant-stand* to which the instrument can be attached, and which is so arranged that the sextant can be placed in any required plane and thus firmly held. Unless a wide, firm, and level bed is available for standing the sextant-stand upon, it is somewhat difficult to adjust, and from the difficulty of reading the vernier with the sextant in position for observing, it results that much time is lost in shifting and adjusting the stand, so that, generally, it is not advisable for a novice to further complicate the difficulties of observing, by the use of a stand.

Observing with the Sextant.

Considerable practice is necessary before any facility in observing with this instrument can be obtained. The best position of the observer, arrangement of the mercurial horizon,

position of watch, etc., can only be decided by each individual observer for himself, and after some experience.

Kit required. The instruments and materials required for sextant observations are—

Sextant complete.	Thermometer.
Artificial horizon complete.	Pocket chronometer or watch.
Angle book and pencil.	Lantern for night work.
Barometer.	

Angle Book. The information to be noted in the angle book is as follows:—

1. Nature and purpose of observations.
2. Date and place of observations.
3. Approximate latitude and longitude.
4. Number of sextant used.
5. Index and instrumental errors of sextant, and observations in detail by which the former has been found.
6. Chronometer used, and whether adjusted to S. T., G. M. T., or L. M. T.*
7. Chronometer rate and correction.
8. Readings of barometer and thermometer.
9. Name of object and the limb observed.
10. Its position E. or W., or, if passing the meridian, N. or S. of zenith.
11. Observed angles.
12. Observed times.
13. State of weather and sky.
14. Statement whether the observations are good or not.

Everything should be noted which affects an observation and should be clearly recorded, *so that it can be worked out by an independent person*, if necessary.

All information possible, as detailed above, should be entered before beginning to observe.

It may be laid down, *as a general rule*, that no astronomical observation should ever be taken without the chronometer time of the observation being also noted.

Taking Time. For noting the actual moment of contact, when there is no assistant for taking time, first count the number of beats the watch makes to one minute, and from this deduce the value of any convenient number of beats (say 5 beats equal to 2 seconds). Place the watch in a position where it can be easily seen, and the beats distinctly heard. *From the moment of contact* begin to count the beats (0, 1, 2, etc.)—turning the head at the same time towards the watch—and note the time indicated, say at the fifth beat, then this time, less 2 seconds, gives the time of observation.

With an assistant to book the observations, the work is much facilitated. Just before contact is made, the observer gives the caution to his assistant, 'Are you ready?' and at the exact moment of contact he says 'Now,' for the assistant to note and write down the time. The angle is then read and written down, and the next observation proceeded with.

* *Vide* p. 280.

THE SIX-INCH TRANSIT THEODOLITE.

The Six-inch Transit Theodolite.

A general description of the transit theodolite has been given in Chapter III., Part I., so that it is only necessary to treat here of its use as an instrument for taking astronomical observations.

For the above purpose the 6-inch is the most suitable size, though the 5-inch lighter instrument is sometimes preferred. With observations taken with this instrument, very satisfactory determinations of latitude, time, and azimuth, can readily be arrived at.

The axis of the instrument is usually perforated, and a lamp supplied wherewith to illuminate the field of view, but if not perforated, the field can be illuminated by directing the rays of a lamp on to a strip of paper, or a metal reflector, fixed to the object end of the telescope, and bent to an angle of 45° with the axis.

When taking a 'set' of observations, the telescope should be reversed between each pair of readings, in order to eliminate any residual error after adjusting for 'collimation.' (Face right and face left, *Vide* Part I., p. 192).

Instructions for Observing with the Transit Theodolite.

The following instructions for observing with the transit theodolite may prove useful:—

1. In observing with a theodolite it is necessary that the image should be well defined and yet so faint as not to fatigue the eye.

2. The elimination of parallax must be carefully attended to, as in reversing the telescope the eye-piece is apt to get slightly out of adjustment.

3. When observing the sun—
   - (*a*) A dark glass must be used over the eye-piece.*
   - (*b*) Accurate observations of both limbs must be taken, leading and following limbs for azimuth, upper and lower limbs for altitude.
   - (*c*) Observation is made by the contact of a limb of the sun with the centre vertical or horizontal wire.

4. When observing a star—
   - (*a*) Only sufficient light should be thrown upon the cross-wires to show them up distinctly without obscuring the star. For this purpose the illuminating lamp may be used, or the paper reflector on the object end of the telescope.
   - (*b*) Contacts are made at the intersection of the centre wires only.

5. In finding time by a transit over the meridian, observations on all the vertical wires are made.

6. When reading the vertical circle, index errors are eliminated by reversing the telescope, between each pair of readings, and taking half the readings with 'circle (or face) right' and half with 'circle (or face) left.'

7. It is not practically advisable to expend time in bringing the vertical circle bubble to the exact centre of its run for each observation, but it is sufficient if both ends of the bubble are within the scale marked on its upper surface.

* The surveyor may have to use a theodolite, unprovided with a dark-glass for the eye-piece. In such a case, one of the writers has successfully resorted to the following expedient, for Sun observations. A disc of cardboard was fitted inside the dew-cap. In the exact centre of this disc, a small hole was pierced, and carefully reamed out with a pencil cut to a pyramidal form. A three-quarter inch objective, rather less than one-eighth of an inch in diameter, gave an image of the Sun, which could be comfortably observed. If this device is used, precept (*b*) is important on account of 'aberration.'

Level Correction.

The positions of the two ends of the bubble are noted *before* each reading of the vertical arc, and the value of the different divisions of the scale being known (*see* following), a correction is applied, according to the distance of the centre of the bubble from the centre of the scale. The reading of the bubble is registered as 'object-end' O, or 'eye-end' E, according as the end is near the object-glass or eye-piece. If the O readings be greater than the E, the correction is *added* to the altitude on the vertical arc, and *vice versâ.*

Example.—

O Readings.	E Readings.
14	15
13½	15½
17	12
15	14
59½	56½

Excess of O readings over E readings $= \frac{3 \text{ graduations}}{4}$

Centre of bubble is therefore $\frac{1 \cdot 5 \text{ graduations}}{4}$ towards O, and the correction is positive.

Value of graduation of bubble, say $= 12''$

Correction for bubble readings therefore $= + \left(12'' \times \frac{1 \cdot 5}{4}\right) = + 4'' \cdot 5$

Let L = Level correction.
n = No. of observations.
V = Value of one division of the scale (ascertained specially for each instrument).

Then
$$L = \frac{O - E}{2n} \times V$$

To find Value of Level Divisions.

The value of a division can be determined by attaching the level tube to a vertical circle, and noting the number of seconds on the circle corresponding to a motion (of the circle and level together) which carries the bubble over a given number of divisions.

Thus, suppose we read the O and E ends of a level thus attached to a circle, and also read the circle itself as follows:—

	O	E	Vertical Circle
1st reading . .	5·5	14·5	1° 40′ 49″
2nd „ . .	17·5	2·0	1° 43′ 53″
∴ Difference . .	12	12·5	0° 3′ 4″

∴ (Mean) $12 \cdot 25 \times V = 3' \; 4''$

∴ $$V = \frac{3' \; 4''}{12 \cdot 25} = 15'' \cdot 02^{*}$$

* Instrument-makers also use a stout metal bar having two points at one extremity, and a pointed micrometer-screw at the other end. The level-tube is attached to the bar, with wax, or otherwise. The whole is placed on a levelled surface-plate. Knowing the distance between the two points at one end and the point of the micrometer-screw, and the value of one turn of the screw, the value of V can be determined with ease.

The determination of V for the level attached to the vertical circle of a theodolite can also be obtained as follows, without the use of the vertical circle belonging to a separate instrument :—

Direct the telescope on a distant but well-defined object. Clamp the vertical circle, and read both ends of the bubble and the vertical circle. By *one of the foot-screws*, elevate or depress the telescope until the bubble has travelled over a given number of divisions. Having again read both ends of the bubble, direct the telescope again on the object by means of the vertical circle tangent screw, and read the vertical circle. We then have the same data as above for determining V.

Foundation for Theodolite. It is most essential that the theodolite should be supported on a very solid foundation. The nature of the ground affects the method of doing this. Generally, it is sufficient to drive strong stakes as far as possible into the earth, and then to cut them off level with the surface, to form an immediate support for the stand of the instrument.

Kit Required. The instruments and materials required for theodolite observations are :—

A theodolite complete with stand.	Thermometer.
Angle book and pencil.	Pocket chronometer or watch.
Barometer.	Lantern for night work.

Angle Book. The information to be noted in the angle book is as follows :—

1. Nature and purpose of observations.
2. Date and place of observations.
3. Approximate latitude and longitude.
4. Number of theodolite used.
5. Vertical and horizontal collimation-errors, and position of vertical circle for which the algebraic signs hold. Face right or face left.
6. Chronometer used, and whether adjusted to S.T., G.M.T., or L.M.T.*
7. Chronometer rate and correction.
8. Readings of barometer and thermometer.
9. Name of object and limb observed.
10. Its position E. or W.; or, if passing the meridian, N. or S. of zenith.
11. R. O. (Referring Object) used, and its horizontal readings.
12. Observed times.
13. Level readings.
14. Observed altitudes.
15. Horizontal angles of object observed.
16. State of weather and sky.
17. Statement whether observations are good or not.

Everything should be noted which affects an observation, and *should be carefully recorded*, so that it can be worked out *by an independent person*, if necessary.

All information possible, as detailed above, should be entered before beginning to observe.

* *Vide* p. 280.

The method of timing an observation is similar to that described for the sextant.

Theodolite and Sextant Compared.

A transit theodolite possesses the following advantages over the sextant for astronomical observations.

(1) It measures horizontal angles directly, and a round of several angles can be measured with less trouble than with a sextant.

(2) It measures small vertical angles of elevation or depression, which could not be measured with an artificial horizon.

(3) Its telescopic power is usually higher than that of the sextant.

(4) It can be manipulated so as to eliminate instrumental errors such as eccentricity, collimation, and index errors.

The disadvantages of the theodolite are to be found in its greater cost, bulk, and weight. For astronomical observations, where altitudes only have to be observed, the sextant is to be preferred to a small theodolite with 3 to 4 inch circle.

Timekeepers.

For work in the field, a keyless lever watch or semi-chronometer is recommended, such watches being now easily repaired, and less liable to get out of order, than full chronometer watches. They can be carried in the pocket, under conditions of fairly rough usage, and will not fall far short of a chronometer, at least in the regularity of their rates. For convenience in counting, a watch should not beat more than five times in two seconds.

For astronomical purposes, it is necessary to know the error and rate of each clock, chronometer, or watch, at any given date, from which the chronometer correction at any other date can easily be found.

The Error of a Watch.

It is well to mention here that 'the error of a watch' is the amount which it is fast or slow on correct time, and hence a watch *fast* has a (+) error and a watch *slow* has a (−) error.

The Correction to Watch Time.

'The correction to watch time' is the amount to be applied as a correction to the time indicated by a watch to reduce the same to correct time. Therefore

A watch *fast* has a (−) correction
A watch *slow* has a (+) correction

Rate of Watch.

A watch must necessarily *gain* or *lose* on mean time, but though human ingenuity cannot succeed in turning out a watch which can keep absolutely true time, it has succeeded in making clocks, chronometers, and watches, which answer the same purpose by having a very uniform rate. Watch rates are noted as 'gaining' or 'losing,' *usually*, per diem. Chronometers and watches are compared by noting coincident beats.

Semi-Permanent Instruments.

Semi-permanent Instruments.

It is not considered necessary to describe and give the adjustments of semi-permanent instruments used sometimes in the field, such as, the portable transit, the zenith sector, and the altazimuth.

TELEMETERS.

Telemeters. The various instruments which have been devised for rapidly determining distances, without chain measurements, may be classed under two principal heads.

(A) Those in which the base is a rod or staff, held at the point whose distance is to be determined.

(B) Those in which the base is measured or fixed at the point of observation. Most military range-finders belong to this category.

Class A may be subdivided into two sub-classes:—

(*a*) Those in which the distance is determined by the length on a staff which subtends a constant angle.

(*b*) Those in which the distance is determined by the angle which a constant length on a staff subtends.

The Principles Involved. The principles on which instruments belonging to class (*a*) depend may be illustrated by the following simple arrangement (*vide* fig. 4).

AB is an ordinary sight-rule.

The back-sight B is pierced with a small eye-hole O. The fore-sight A is provided with two horizontal wires C and D equidistant from a central wire E. The distance between the two wires is made some aliquot part of the distance A B such as one-tenth or one-twentieth. Suppose now that a graduated staff S is held up at some given distance. Looking through the hole at O the graduation intersected by the upper and lower wires, respectively, are read off. The difference is the distance F G subtending the fixed angle G O F. Then

$$FG : CD :: HO : AB$$

$$HO = \frac{FG \times AB}{CD}$$

So that the distance is obtained simply by multiplying the substense on the staff by the constant $\frac{AB}{CD}$.

This simple form of telemeter, though sometimes useful for rough work, is of limited applicability, owing to the short distance at which the graduations of a staff can be read with the naked eye.

The Telemetric Telescope. The same principle may be applied to a telescope by providing two horizontal, or vertical wires, as in the tacheometer.

When, however, a telescope is used, the relation between the subtended distance on the staff and the horizontal distance, is less simple than in the case of the elementary instrument, just described, because the distance between the object-glass and the cross-wires is not constant.

The theory of the telemetric telescope is as follows:—

Theory of the Telemetric Telescope. Let O O (fig. 5) be the object-glass of the telescope, C and D two cross-wires fixed in the diaphragm, S S a graduated staff, held at right angles to the optical axis of the telescope.

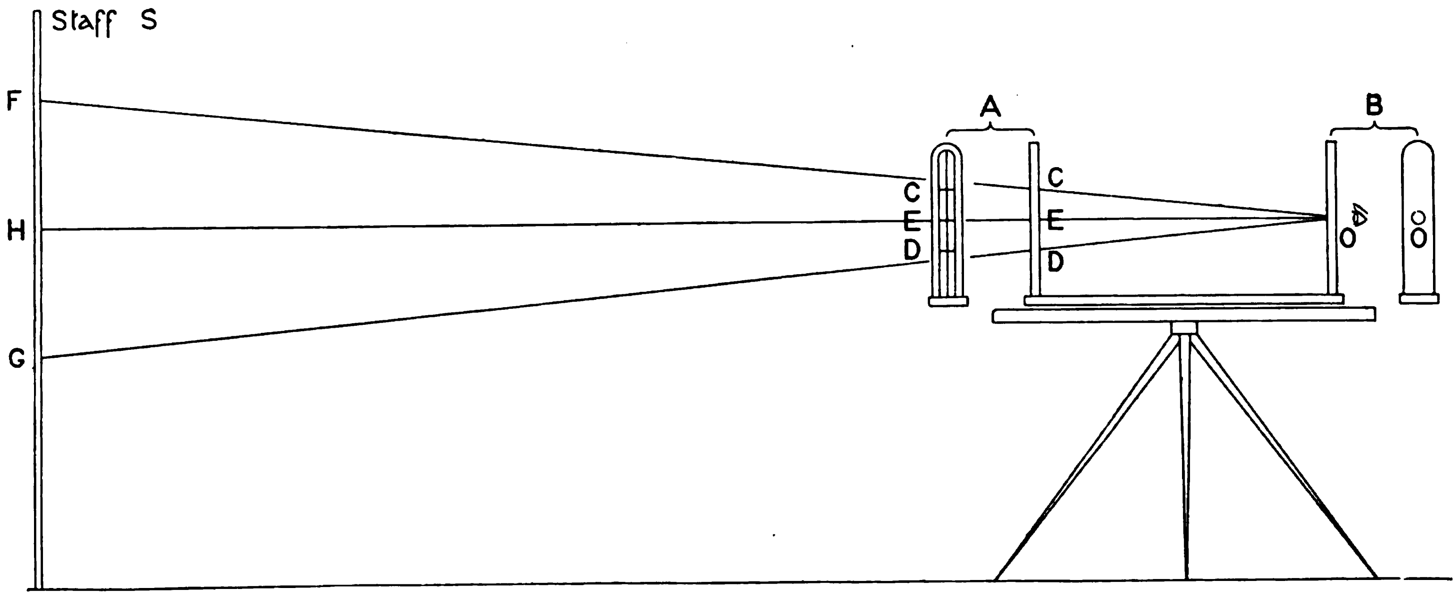


FIG. 4.

Now,

Let x = The distance from staff to the object-glass of the telescope.

l = The distance from object-glass to the cross-hairs, when properly focussed on the staff.

s = The distance visually intercepted between the top and bottom wires on the staff, that is, the difference between the staff readings of the upper and lower wires. This may be called the 'subtense-distance.'

f = The focal length of the object-glass, that is, the distance from object-glass to cross-hairs, when focussed on some very distant object, such as the Sun.*

d = The distance between the two cross-wires C and D.

Then, by the principles of optics,

$$\frac{x}{l} = \frac{s}{d} \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{f} = \frac{1}{x} + \frac{1}{l} \quad . \quad . \quad (2)$$

From equation (2)

$$l = \frac{f\,x}{x - f}$$

Introduce this value of l in (1), whence,

$$x = \frac{f}{d}\,s + f \,. \quad . \quad (3)$$

But f and d are both fixed quantities. The fraction $\frac{f}{d}$ is usually made some round number, say 100. In this case then, the distance of the staff from a point one focal length in front of the object-glass, is obtained by multiplying the subtended

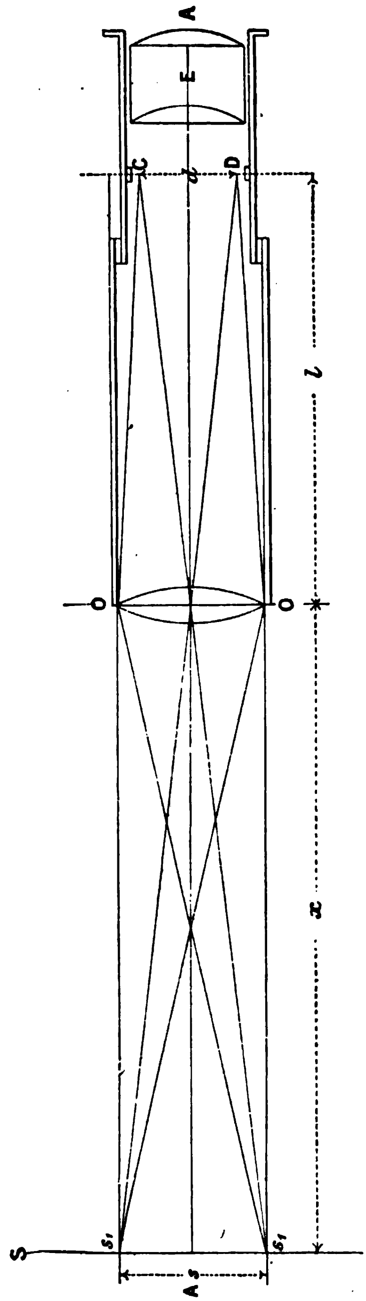


FIG. 5.

* Note: f is called the *solar focal distance* of the objective, because it is the distance at which rays practically parallel, like those from a point on the Sun's Disc, or from a Fixed Star, practically unite in a point or focus, after passing through the lens.

distance on the staff by 100, or whatever may be the coefficient of the instrument, namely the fraction $\frac{f}{d}$.

To obtain the distance from the centre of the instrument, add to the subtense distance on the staff, multiplied by 100, a 'constant,' namely the 'focal length of the object-glass, added to the distance of the object-glass from the centre of the instrument.' This 'constant' will be called the 'focal correction,' and will be represented by k. The constant to be added is usually about $1\frac{1}{2} f$ with ordinary instruments. It may, however, be determined by actual measurement, thus:—Focus the telescope on some distant object, and carefully measure the distance from the object-glass to the cross-wires (i.e. from the centre of object-lens to the heads of the screws that hold the diaphragm), as well as from the middle of the object-glass to the centre of the instrument.

The sum of these distances will be the correction to be added.

Distance at which Solar Focus is Obtained.

With ordinary telescopes, the distance at which solar focus is obtained, for all practical purposes, is not very great. Take, for example, a telescope having a focal length of one foot. Assume it to be focussed on a point 5000 feet distant.

$$\frac{1}{f} = \frac{1}{5000} + \frac{1}{l}$$

Now $f = 1$

$$\therefore \quad \frac{1}{l} = 1 - \frac{1}{5000}$$

and $l = 0 \cdot 9998$

but $f = 1 \cdot 0000$

$$\therefore \text{ difference } = 0 \cdot 0002 \text{ of a foot}$$

$$\text{or } 0 \cdot 0024 \text{ of an inch}$$

So then, with a 12-inch telescope, at about one mile, the actual focal length differs, from the solar, by about one four-hundreth of an inch only.

The Anallatic Lens.

In some telemetric telescopes, an 'anallatic' lens is added, which has the effect of eliminating the 'constant correction k,' above referred to, so that the distance read is measured to the centre of the instrument. The effect of this lens is discussed in Chapter II., Part I., p. 58.

It is questionable whether the complication introduced, and the loss of light caused by the introduction of this lens, is justified by the slight saving in labour which it effects.

Adjustment of the Subtense Wires.

In some telemetric telescopes, a means of adjusting the distance between the subtense-wires, and between them and the centre-wire is provided, independently of the usual screws which bring the 'centre wire' into the 'optical axis' of the telescope. In this case the adjustment is effected as follows. First, adjust the centre wires, both 'vertical' and 'horizontal,' for collimation, in the ordinary manner. If the

instrument has no anallatic lens, determine the constant focal correction in the manner described. Suppose that the focal constant is 1·5 of a foot. Chain out 101·5 feet or 201·5 (generally $n + k$ feet), from the *centre* of the instrument, and set up a staff at the point set out. Level the telescope and direct it to the staff, focussing carefully. The three wires should now subtend equal spaces on the staff, and the difference between the reading of the upper and lower wires should be exactly $\frac{n}{100}$, that is, 'the chained distance less the constant focal correction.' For example, if the staff were put at 101·5 feet, the upper and lower wires should subtend 1 foot on the staff. If the staff were at 201·5 feet, then 2 feet should be subtended, and so on.

If this be not the case, move the upper and lower wires, by means of the screws controlling them (leaving the centre wire untouched), until they are equidistant from the centre wire and subtend the proper space on the staff.

If the instrument be provided with an anallatic lens, the procedure is the same, only no correction is added, and the round distance is set out.

For adjusting or checking, it is convenient to use a drawing-scale, attached to a wall or door at a moderate distance from the instrument. The telescope being levelled, the scale is moved up or down by an assistant, until some round division is intersected by the horizontal centre-wire. The scale is then fixed, by means of drawing pins, and the adjustment is made.

The Glass Diaphragm.

The most practical plan, however, is to use a glass diaphragm, with the subtense lines engraved upon it, at the proper distance apart, in lieu of adjustable cross-wires.

A skilful instrument maker can engrave the lines with a *probable* error, less than the *unavoidable* error of observation. If the glass diaphragm be used, it is well to add a pair of vertical subtense wires, so that distances can be read on a staff held in a horizontal position. It is well also to add two diagonal lines to mark the central point. Without these, it is very possible to use the wrong intersection when observing in a bad light or to an object partially obscured by foliage.

With the glass diaphragm, the only adjustment required is the ordinary collimation adjustment of the central intersection.

The accuracy of the subtense line should be checked in the manner already described. If there be any appreciable error, it will be constant, and may be allowed for.

The Tacheometer.

The tacheometer, as originally invented by Green in 1778, was in reality a theodolite, having a pair of lines ruled upon glass parallel to, and one above, and the other below, the axial horizontal line of the diaphragm, and at such a distance apart as to subtend a length, measured on a divided staff, equal to a certain part, commonly $\frac{1}{100}$, of the distance of the staff from the centre of the instrument. An alternative arrangement of hairs or points, attached to movable diaphragms, allowed of an adjustment of the distance between the subtense wires by means of a micrometer screw.

This instrument, however, was not really exact, inasmuch as no allowance was made for the correction (above referred to), to be added to the apparent distance

registered by the subtense lines or hairs, on account of the refraction in the telescope, and the consequent varying position of the focus.

In order to obtain the desired constant focus, and exact measurement from the vertical axis of the instrument, recourse must be had to the anallatic telescope invented by Professor Porro of Milan, and described in Chapter II., Part I., p. 58.

In order to compensate for the considerable loss of light occasioned by the employment of the additional lens, the telescopes of tacheometers are necessarily made of large size, and the eyepieces are usually of greater magnifying power than those of ordinary theodolites.

When the telescope is in a horizontal position, the distances may be read directly, thus, if the reading of the lower line be 0·25 on the staff while the upper line reads 1·87, the number of units intercepted is 162 and the distance of the staff from the centre of the instrument is 16,200 units or 162 links, feet, yards, or metres, according as the unit used is $\frac{1}{100}$ part of either one of these measurements.

If the telescope be inclined so that the line of sight falls above or below the horizontal, the line drawn through the centre of the instrument to the staff is longer than it would be were the telescope horizontal, also the number of units intercepted is greater than would be the case if the telescope were horizontal, in proportion to the angle of inclination. The number of divisions must therefore be reduced by multiplying them by the cosine of the 'angle of elevation,' and the 'distance' thus obtained, which is that along the angle of inclination, must in like manner be multiplied into the cosine of that angle.

It is usual to take the distance on the slope as calculated directly from the interceptions without reduction, and to multiply the result into the cosine *squared*, which will give the same figure as that arrived at by the more detailed method above indicated.

If the angle of inclination of the telescope with the horizontal be 3° while the units intercepted number 162, the horizontal distance will be 16,155 units or say 161·55 feet.

Up to an angle of elevation or depression of 1° the correction may be ignored, except when the utmost accuracy is required, the error only amounting to $\frac{3}{20}$ of a foot in 500 feet, while the errors due to defective observation alone, are far greater than this dimension. A full description of these corrections is given under 'Telemeter.'

The horizontal and vertical circles are usually divided centesimally, a method which permits of the use of the slide rule, and is preferred on the Continent, but has not found favour in England, nor does it offer all the advantages claimed for it, which would be better provided for by a division of the circle into 360 degrees, and each degree into 100 parts.*

Several devices have, at different times, been introduced as attachments to the tacheometer, with the object of mechanically performing the reduction of the inclined distance to the horizontal plane, and at the same time obtaining the difference of level between the instrument and the staff station.

* Kern, of Aarau (Switzerland) supplies a slide-rule for effecting reductions to the horizontal, and also calculation of differences of level which is adapted to the ordinary sexagesimal notation.

In the Wagner-Fennel tacheometer, a scale is fixed alongside the telescope, moving with it, and a slider attachment is moved along this scale to the point corresponding with the direct distance, ascertained by a stadium-rod held at right-angles to the line of sight. The same slider is clipped round a vertical scale in such a way as to permit of an up-and-down motion, while the scale itself travels to and fro sideways along a third scale, fixed to the upper limb of the instrument. The dividing is so arranged, on the three scales, that the horizontal distance is indicated on the horizontal scale, while the vertical travelling scale gives the height of the intersection of the central line of sight upon the stadium, above the centre of the axis of the telescope.

This apparatus adds greatly to the weight of the instrument, and is necessarily awkward to handle, while the accuracy obtainable depends entirely upon the closeness with which the scales can be read in the field, and the truth of their divisions and alignment.

Moreover, the number of the parts is considerable, their delicacy extreme, and they are very liable to injury, and to disturbance in their adjustment.

The Cleps. The cleps was invented by Professor Porro, of Milan. It differs considerably from the tacheometer. While the latter has its 'horizontal' and 'vertical' circles uncovered, as in the ordinary theodolite, in the former they are enclosed in a cubical box. In the cleps, instead of the angle being read by means of verniers, micrometer wires applied to the microscopes are used. The telescope of the tacheometer is 'concentric,' that of the cleps is 'eccentric.' The graduation of the horizontal circle in the tacheometer proceeds in the same direction as the hands of a watch, whereas in the cleps it is graduated in the opposite direction. The tacheometer is provided with an ordinary Ramsden eyepiece, the cleps, on the other hand, has a multiple Argo eyepiece. Both instruments have anallatic lenses, and in both the circles are divided centesimally. Porro used a staff of triangular shape, measuring $2\frac{1}{2}$ inches on each face, and graduated on all three faces. There are different graduations on each of the sides, one with very fine lines for near readings, one in equal blacks and whites, and one divided to metres and decimetres only, for distant readings.

This instrument was used on the Mont Cenis Tunnel Survey. The degree of accuracy obtained was so great that the error never exceeded $\frac{1}{2000}$.

The disadvantages attached to it are its extreme delicacy, and the very careful handling consequently required, the impossibility of testing its adjustments, and the eccentricity of the telescope, which amounts to nearly 3 inches in the larger instruments. The small size of the circle is objectionable, as also are the shadows cast by the windows used for lighting the circles, and which are apt to cause false readings.

The Subtense Theodolite. We turn now to the second division of telemetric instruments, viz. those in which a 'fixed base' and 'variable angle' are used.

The subtense-theodolite is an example of such an instrument. It consists of an ordinary theodolite, provided (in lieu of the diaphragm) with a micrometer, in the common focus of the object- and eye-glasses.

The micrometer consists of an arrangement by which a pair of wires can be moved across the field of view by means of a screw, drawing them near to, or separating them from, the fixed central wire.

Each micrometer wire is moved by a fine screw, the 'drum-head' at the end of which is divided into one hundred parts. The distance from the centre wire to either of the movable wires is therefore measured in terms of the pitch of the screw, or number of turns, and hundredth of a turn, made by the screw in moving the wire, so as to intersect any object within the field of view. The number of whole turns made are counted by means of a comb, fixed in the field of view, the parts of a turn are measured by the graduations on the drum-heads serving to turn the screw. The sum of the distances from the centre wire, is the distance between the two subtense-wires, expressed in turns of a revolution of the micrometer screw.

Usually, the 'micrometer' is made so that it may be turned round, and used either 'vertically' or 'horizontally,' as most convenient.

The 'micrometer' therefore measures the distance between the 'subtense wires,' when they intersect two points on a staff, at a known and fixed distance apart.

The theory of the 'subtense theodolite' is precisely the same as that of the 'telemeter telescope' already described, and the same equation applies, viz.

$$x = \frac{fs}{d} + k$$

where

f = the focal length of the telescope
s = the length of the base
d = the distance between the wires
k = the focal connection, which is not required if the telescope has an anallatic lens
x = the distance from the instrument to the staff.

Now, let n be the number of turns and parts of a turn, made by both the 'micrometer screws,' when the 'subtense wire' intersect the marks at the extremities of the staff, and t the value of one turn in feet. Then

$$x = \frac{fs}{nt} + k$$

The value of f may be determined, with sufficient accuracy for the purpose of finding k, in the manner as already described. Not so for f in the factor $\frac{fs}{nt}$. It is not however, necessary to know f absolutely, since all that is required is to ascertain the value of $\frac{f}{t}$. This can be done experimentally as follows:—

Chain out with all possible accuracy, some distance, say $(500 + k)$ feet, from the centre of the instrument, and at that distance set up a graduated staff accurately 'vertical.' Level the telescope, and take the staff reading of the centre wire. Now move the wires, by means of the micrometer screws, until one wire intersects the staff exactly five feet above, and the other, five feet below the inter-

section of the centre wire. Count the turns of each micrometer screw, and read off the fractions of a division on the heads.

Suppose that the following staff readings were observed:

Upper wire	10·73
Middle wire	5·73
Bottom wire	0·73

Micrometer.

Right-hand wire	6·000
Left-hand wire	6·000
	12·000

Let $k = 1·5$

and $s = 10$, the length on the base of the staff.

Then

$$501·5 = \frac{fs}{12·000 \times t} + 1·5$$

$$\therefore \quad \frac{f}{t} = \frac{500 \times 12·000}{10} = 600$$

then 600 is the coefficient of the instrument. Distances are then obtained by the expression,

$$x = \frac{6000}{n} + k \text{ (when } s = 10\text{)}.$$

The above case represents that of an instrument having a focal length of one foot, exactly, and a micrometer screw with fifty threads to one inch, so that $\frac{f}{t} = 600$. So exact a proportion would hardly ever obtain in practice.

It is evident that each turn of the micrometer screw should move the wire through an equal space on the staff (when it is perpendicular to the optical axis of the telescope). If it do not, then either the screw is not correctly cut, the pitch varying at different points, or there is an index-error.

Determination of Errors by Centre Wire Intersections.

By intersecting points successively, say 0·5, 1·0, 1·5, etc. feet *above* and *below* the point of intersection of the centre wire, this index error, and error of the screw, may be determined. Suppose that the staff-readings were

Staff Readings.	Upper Micrometer Wire Readings.
5·37	zero
6·37	1·20
7·37	2·40
8·37	3·60
9·37	4·80
10·37	6·00

then the micrometer is correct.

If, on the other hand, the micrometer readings run thus:—

1·2003
2·4003
3·6003

then the micrometer screw is 'good,' but an 'index error' of 0·0003 exists in this case, to be deducted from all readings.

If the readings be irregular, then the micrometer screw is bad, and if the irregularities be considerable, the micrometer should be returned to the maker. If this cannot be done, or if the irregularities be but slight, the observations should be carefully repeated, a mean value taken, and the results tabulated. A table should then be prepared, showing the co-efficient to be used for one, two, or three, etc. turns of the screw.

If the micrometers can be placed either 'horizontally' or 'vertically,' the staff may be held either horizontally or vertically.

In many cases, the horizontal staff will be most convenient, inasmuch as a longer staff may be used in this position than could be conveniently held vertically. In the horizontal position, a staff 20 feet long may be used, and might be supported upon a pair of light tripods. The staff need not be graduated, but should then have discs attached to it.

The computation of distances may be performed by means of a table of reciprocals, or by logarithms, the equation being

$$x = \frac{C\,s}{n} + k$$

Example.—

Let $C = 600$

and $s = \text{the base} = 10$

Right-hand micrometer, reading	5·632
Left-hand " "	5·634
Sum of readings $= r$	11·266
Reciprocal of sum is	·088763
$x = 600 \times 10 \times {\cdot}088763$ . . .	532·57
Add k	1·50
Horizontal distance x is	534·07

Or, by logarithms,

log n = log 11·266 . . .	1·05177
colog n	$\bar{2}$·94823
log 600 × 10	3·77815
Log apparent distance . = log 532·57 =	2·72638
Add k . . . . . 1·50	
Distance . . . . = 534·07	

The reduction to the horizontal is performed in the same manner as in the case of the telemetric telescope.

With the *horizontal* staff, the 'apparent distance' is augmented by the focal correction, and multiplied by the cosine of the 'vertical angle.'

With the *vertical* staff, the 'apparent distance' is multiplied by the cosine of the 'vertical angle.' The product, augmented by the focal correction, is the 'oblique distance,' which again multiplied by the cosine of the 'vertical angle' is the 'horizontal distance.'

A high degree of accuracy, with great rapidity, can be attained by means of the subtense theodolite.

Compared with the 'telemetric telescope,' the 'subtense theodolite' will doubtless give the better results. It is more easy to intersect a well defined object with a movable wire, than to read a graduated staff. On the other hand, to obtain even the apparent distance, with the subtense instrument, calculation is required, whereas with the telemetric telescope, an apparent distance is read at sight—a determination which, under many conditions (for example, moderate vertical angles) may be sufficiently accurate for all practical purposes.

The Omnimeter.

The omnimeter, by Scholdt, is an instrument which may either be used with a fixed base, or with a graduated staff.

This instrument is much used in India and America, and consists of an ordinary theodolite, with a powerful microscope, furnished with cross wires in the focus, attached to the telescope axis, and at right angles to its optical axis. By means of this microscope, the graduations of a horizontal scale, fixed on the upper plate of the theodolite, and having a small motion in the direction of the plane on which the telescope revolves, may be read with precision.

The scale is 4 inches long, and is divided into 100 primary divisions, which are again subdivided, making 200 divisions in all.

The micrometer screw has 50 threads per inch, and the head is divided into 100 divisions, and these can be subdivided by the vernier five times. The total subdivision thus becomes 100,000 in 4 inches, and when the telescope is level, the microscope should read 50,000 on the scale, i.e. the central division.

The distance from the centre of the axis of the telescope to the tangent-surface of the scale is 6 inches.

The distances may then be obtained by the following geometrical principle. First, when a base of constant length is used.

Let O (fig. 6) be the centre of the telescope axis. Let O Z be the optical axis of the microscope, which is accurately and rigidly fixed at right angles to the line of collimation of the telescope. Let $S S^1$ be the graduated scale, which must be accurately parallel to the line of collimation of the telescope, when the latter is horizontal. The zero Z of the scale must be in the line of collimation of the microscope, when it is vertical. Let O C = x, the desired horizontal distance.

Let the distance O Z = h, a constant quantity. Let $\frac{1}{C}$ = the ratio of the length of one graduation of the scale, to h the height of the axis above the scale of the telescope.

A base staff S, with marks A and B near to its extremities, is held perpendicularly, at the point whose distance is to be determined. The mark B is now intersected by the horizontal wire of the telescope. The microscope is now focussed to the horizontal scale, and the division of the scale, intersected by the microscope wire, is read off. To enable the fraction of a scale-division to be read with precision, a mircrometer screw is provided, by which the scale can be moved horizontally. The pitch of the screw is such, that one turn moves the scale through a space of one division. The head of the screw is divided into one hundred parts. When, therefore, the cross-wire of the microscope falls between two divisions of the scale, the micrometer is turned until the division

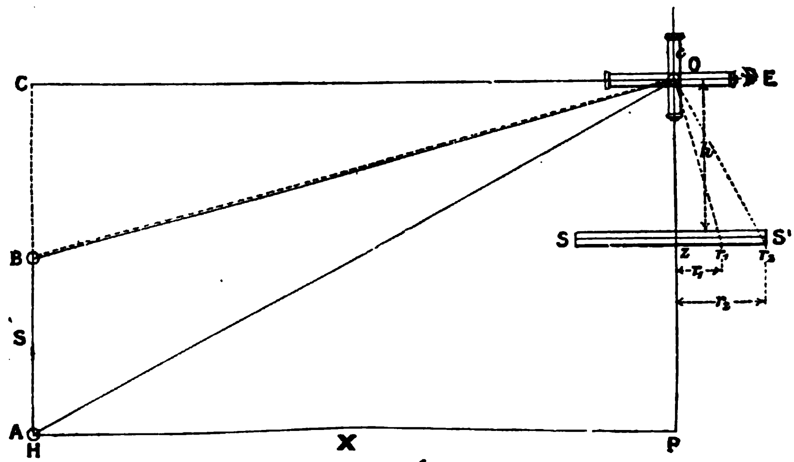


FIG. 6.

next below the wire is brought into coincidence with the wire. The unit division is then brought into coincidence with the wire. The whole divisions are read off by the microscope.

Call the first reading r_1, next, let the point A be intersected by the horizontal wire of the telescope, and a second scale reading r_2 obtained.

Then, since the optical axes of the telescope are always perpendicular to each other, it is evident, by similar triangles, that

$$\frac{BC}{CO} = \frac{r_1}{h} \qquad BC = CO\frac{r_1}{h}$$

$$\frac{AC}{CO} = \frac{r_2}{h} \qquad AC = CO\frac{r_2}{h}$$

$$AC \pm BC = x \times \frac{r_2 \pm r_1}{h}$$

$$\therefore \quad x = \frac{hs}{r_2 \pm r_1}$$

where x is the distance, and s the length of the staff.

The distance can be computed by means of a table of reciprocals, using the

plus sign, when one angle is an elevation and the other a depression, and the minus sign, when both are depressions or both elevations.

The divisions of the scale are so selected as to be some aliquot part of the distance h, from upper plate to collimation axis, usually $\frac{1}{100}$. In any case, r may be written $\frac{h}{n\,C}$ where $\frac{h}{C}$ is a constant, and $n =$ the number of divisions and parts of a division, so that

$$x = \frac{h\,s}{\frac{h}{C}(n_2 \pm n_1)}$$

$$= \frac{C\,s}{n_2 \pm n_1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Suppose $\quad C = 100{\cdot}0$

and $\quad s = 10{\cdot}0$

then $\quad x = \frac{1000}{n_2 - n_1} \quad . \quad . \quad . \quad . \quad . \quad (2)$

Horizontal distances may be read directly, by means of a graduated staff, as follows:—

Direct the telescope to some even division of the graduated staff. Move the scale, by means of the micrometer screw, until the cross-hair of the microscope intersects exactly some one division of the scale.

Without moving the micrometer, move the telescope by means of the tangent screw, until the cross-hair of the microscope exactly intersects some other division of the scale, distant n divisions from that first intersected. Read the staff again. The difference of the two staff readings being called s, the distance is obtained as follows:—

$$x = \frac{C\,s}{n}$$

Suppose that $\quad C = 100$

and $\quad n = 1$

then $\quad x = 100\,s$

Differences of level may be obtained by means of the omnimeter. It is evident that the scale-readings are the tangents of the angles of elevations, to base $O\,Z = h$.

Consequently, the difference of level between O and B is

$$\Delta h = x\,\frac{r_1}{h} = x\,\frac{\frac{h}{n\,C}}{h}$$

$$= \frac{x}{n\,C}$$

$$= \frac{C\,s}{(n_2 \pm n_1)\,n_1\,C}$$

$$= \frac{s}{(n_2 \pm n_1)\,n_1}$$

Adjustment of the Omnimeter.

Omnimeters vary considerably in construction, and consequently no general rules can be laid down for their adjustment.

In addition to the ordinary adjustments, common to all theodolites, the following points have to be attended to.

(1) The 'zero' of the 'horizontal scale,' when the instrument is levelled, should be perpendicularly below the axis of the telescope.

(2) The 'optical axis' of the 'microscope' should be accurately at right angles to the 'optical axis' of the 'telescope.'

(3) The co-efficient of the instrument should be checked.

The last adjustment can be performed as follows:—

Set the instrument up, opposite to a vertical wall, and level it carefully. It should be as near to the wall as is consistent with distinct focussing.

Level the telescope and make a fine mark on the wall at O (fig. 7) to coincide with the horizontal wire of the telescope.

Now make three equidistant marks a, b, and c above O and a_1, b_1 and c_1, below O. Intersect these marks in succession both above and below O, and take the scale-reading for each.

If the adjustments are correct the difference of the scale-readings will be equal.

If not, then there may be an error either in the zero of the scale or in the perpendicularity of the microscope axis, or both.

If the difference of the scale readings for O a and O a_1 differ, but the differences for $a\,b$, $b\,c$, $a_1\,b_1$, and $b_1\,c_1$, are equal, then there is an index error of the scale, but if the differences for $a\,b$, $b\,c$, $a_1\,b_1$, $b_1\,c_1$, differ from each other, then the microscope axis is not perpendicular to the telescope axis.

Thus, if the difference for $a\,b$ is *less* than the difference for $b\,c$, then the axis of the microscope makes less than a right angle, measured on the side of the object-glass of the telescope.

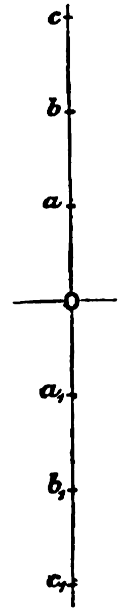


FIG. 7.

The co-efficient of the instrument is determined by observation to a base placed at a measured distance. It is convenient that the co-efficient should be a round number such as 100, so as to facilitate the construction of a reduction table. If on trial the co-efficient is not found to be exactly correct, then the standard table can still be used, by lengthening or shortening the base, as described in connection with the subtense-theodolite.

Appendix F describes a method of determining horizontal and vertical distances with an ordinary theodolite, and a fixed base marked on a staff, due to Mr. Wilfred Airy, M.Inst.C.E.

CHAPTER III.

TACHEOMETRIC SURVEYING.

Definition of the term.

TACHEOMETRIC Surveying, or shortly 'Tacheometry,' is the science of surveying and levelling simultaneously, by means of angular measurements, defined by hairs or lines placed in the focal plane of an optical instrument, and a base read on a staff employed along with it.

Optical Instruments used.

The optical instruments used for this purpose, may be roughly divided into two classes as follows: (*a*) those in which the subtense angle (as defined by the distance between the hairs or lines in the diaphragm) is constant, whilst the staff base is variable, and (*b*) those in which the staff base is of a constant length, whilst the subtense angle is variable.

The instruments which belong to these classes are as follows:

(*a*) The Tacheometer.
The Cleps.

(*b*) The Theodolite, with a subtense or micrometer eye-piece.
The Omnimeter (this instrument may be also used with a variable base).

The construction and adjustment of these instruments are described in Chapter II., Part II.

FIELD WORK.

Field Work with the Tacheometer or Cleps.

In the field the methods employed both for the 'tacheometer' and the 'cleps' are practically identical. In important surveys it is usual to have a party of three, consisting of the engineer in charge, an observer at the instrument, and a booker or recorder of the results. It is advantageous to have two or more staff-holders, according to the nature of the ground, for while one staff is being read, the other or others can be moved to new stations. The above practice is entirely opposed to that adopted in England (where the post of honour is at the instrument), but it is nevertheless undoubtedly the right one.

Staff Holding.

It is of great importance that the staff should be held absolutely plumb, for it must not deviate more than $3^\circ 40'$ from the vertical if the error is to be kept below $\frac{1}{500}$, while if the limit of permissible error is taken at $\frac{1}{2000}$ it must not deviate from the vertical more than about $1^\circ 10'$. With this object, it is provided either with a plumb-bob or still better with a circular level, attached to the staff, in such a position as to be constantly under the holder's eye.

When reading, it is desirablo to direct the lower cross hair to a round

number, say, a foot or metre, as the case may be, and as near to zero on the staff. as the nature of the ground will admit. For investigation of the effect of 'staff holding' on result, *vide infra*, p. 35.

Procedure when Surveying with above Instruments.

The instrument having been set up, and its height measured, the staff-holders are sent in succession to all the points to be surveyed, their movements being directed by signals. Each point is given a number, which is entered in the 'field-book' and on a 'sketch plan.' The telescope is directed to the staff at each successive position, and the readings noted in the proper columns. The 'horizontal' and 'vertical angles' are also read and entered. In this manner three dimensions are obtained for each point, and it can then be fixed by polar co-ordinates, and the relative altitudes calculated.

To Connect with an Adjacent Survey.

If it be desired to connect the survey in hand with another, two or more points on each must be fixed. When all the ground to be surveyed cannot be seen from one station, stations must be arranged and connected one with another, as in a trigonometrical survey.

For a Railway Traverse.

For a railway traverse, the general direction of the line having been set out, a strip of sufficient width to allow of lateral displacement, say 400 yards wide, can be surveyed from the line of stations. Marks are fixed about 200 or 300 feet apart, and numbered. For filling in details, points are chosen, wherever there are any decided variations in level. Each staff holder having his place assigned to him, the engineer signals when he is ready to take the reading, and this being done the staff holder moves on to some other selected point. At every fifth or tenth point, the booker gives a double signal to enable the engineer in charge (in case he should be at a distance from the instrument), to check his sketch.

Bearings to be Frequently Checked.

The bearings of lines should be frequently checked by turning the instrument to zero, when the compass should point to the magnetic north (if this is used as the meridian) or to the required number of degrees, minutes, and seconds, east or west of magnetic north, as the case may be.

*** The Field-Book.**

Mr. Brough recommends that the field-book should be ruled with fifteen columns, (1) for station numbers, (2) height of collimation above point levelled, (3) numbers of the points selected, (4 and 5) horizontal and vertical angles, (6) readings of staff, (7) their differences, (8) height of line of collimation on staff by reading from axial hair, (9 to 14) results as calculated, (15) remarks.

The above method was used by Mr. Moinot for about 1000 miles of railway survey in France.

Effects of Staff Holding on Results.

Hitherto it has been assumed that when measuring distances the telescope is 'horizontal,' and the graduated staff 'vertical.' Excepting when the telemeter telescope is used in connection with levelling operations, this will rarely be the case in practice.

Some observers prefer having the staff at right angles (approximately) to the

* *Vide* Appendix M, p. 322.

'optical axis' of the telescope. To this end, a short 'sight-rule' is attached to the side of the staff, and the staff-holder looks through the sight, and inclines the staff until the line of sight of the rule is directed on the telescope.

Again, the graduated staff may be placed in a horizontal position supported on a light tripod, and laid perpendicular to the optical axis by means of a sight-rule as above.

In both cases, the oblique or apparent distance is read off the staff directly, and must be reduced to the horizontal distance, by multiplying by the cosine of the angle of elevation or depression, after adding the focal constant, if there be one.

The more usual and generally preferable practice is to hold the staff vertical, though in this case the reduction to the 'horizontal' is more complicated.

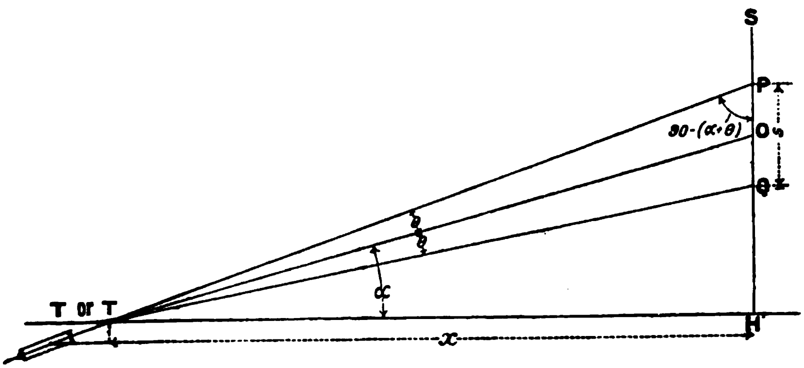


FIG. 8.

Reduction to the Horizontal with Staff held Vertical.

Let T, fig. 8, be the point, one focal length in front of the object-glass, from which the apparent distance is measured, or if the 'anallatic' lens be used, let T be the centre of the instrument.

S H is the vertical staff, P, O, and Q, are the points intersected by the upper, middle, and lower wires, respectively.

Then

Let s = P Q = the subtended distance on staff.

α = the mean angle of elevation or depression.

θ = the angle P T O or O T Q = $\frac{\text{angle P T Q}}{2}$

where the

$<$ P T Q = the angle subtended between the upper and lower wires.

Let x = T H = the horizontal distance.

Then in the oblique triangle T P Q we have

the side P Q = s

" $<$ P T Q = 2θ

" $<$ T P Q = $90 - (\alpha + \theta)$

" $<$ T Q P = $90 + (\alpha - \theta)$

$$\therefore \qquad \frac{TP}{s} = \frac{\sin\{90 + (\alpha - \theta)\}}{\sin 2\theta}$$

$$\therefore \qquad TP = \frac{s\cos(\alpha - \theta)}{\sin 2\theta}$$

But $\sin 2\theta$ is a constant, and therefore we may write

$$\frac{1}{\sin 2\theta} = C$$

$$\therefore \qquad TP = Cs\cos(\alpha - \theta)$$

and

$$TH = x = Cs\cos(\alpha - \theta)\cos(\alpha + \theta)$$

but

$$\cos(\alpha - \theta) = \cos\alpha \,.\, \cos\theta + \sin\alpha \,.\, \sin\theta$$

and

$$\cos(\alpha + \theta) = \cos\alpha \,.\, \cos\theta - \sin\alpha \,.\, \sin\theta$$

therefore

$$x = Cs(\cos^2\alpha \,.\, \cos^2\theta - \sin^2\alpha \,.\, \sin^2\theta)$$

Now θ is a small angle, usually $17' \, 11''$, an angle whose tangent or sine is $\frac{1}{200}$ nearly, and the square of which is $\frac{1}{40000}$ nearly, also $\sin\alpha$ is always less than unity, consequently the second factor is negligible.

Again

$$\cos^2\theta = 1 - \sin^2\theta$$

$$= 1 - \frac{1}{40000}$$

$$= \frac{39999}{40000}$$

$$= 0{\cdot}999975, \text{ nearly.}$$

Therefore, the difference between '$\cos^2\theta$' and 'unity' is also negligible, so that, for all practical purposes,

$$x = Cs\cos^2\alpha \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$x = Cs(1 - \sin^2\alpha) \quad . \quad . \quad . \quad . \quad (2)$$

where C is the coefficient of the instrument, usually 100.

The substitution of unity for $\cos^2\theta$ and the omission of the factor $\sin^2\alpha \sin^2\theta$ amounts to the assumption that the rays TP and TQ are parallel to TO, instead of being inclined towards it at an angle θ.

The equation

$$x = Cs\cos^2\alpha$$

gives the true horizontal distance if the telescope has an anallatic lens

If the telescope be not provided with an anallatic lens, then

$$T\,O = C\,s \cos \alpha + k$$

and

$$x = C\,s \cos^2 \alpha + k \cos \alpha$$

As far as horizontal distances are concerned, k may be substituted for k cos α without any appreciable error. Suppose that $k = 1 \cdot 5$ and that $\alpha = 30°$, a greater angle than should be employed in practice.

$$\begin{aligned} \text{Then} \qquad k \cos \theta &= \cdot 866\, k \\ &= 1 \cdot 299 \\ k &= 1 \cdot 500 \\ \text{Difference} &= 0 \cdot 201 \end{aligned}$$

With smaller vertical angles, the difference will be less.

The factor C s (usually 100 s), will be called the 'apparent distance.' The 'oblique distance' is given by the expression

$$C\,s \cos \alpha + k$$

The true 'horizontal distance' $= C\,s \cos^2 \alpha + k \cos \alpha$

With instruments having an anallatic lens, $k = 0$, so that the expressions become $C\,s \cos \alpha$, and $C\,s \cos^2 \alpha$.

Direct Conversion Scale. Fig. 9 shows a conversion-scale for obtaining the horizontal distances direct, to the scale of the plan. At the foot of the figure a scale of feet is drawn, to the scale of the plan which is being made.

Along the vertical line A B, distances are set off to a large scale, proportional to the *squares of the sines* of the different angles of elevation. In the figure, a scale of 1″ = ·02, was adopted. Thus, to this scale,

$$\text{the length in inches of A . 10} = \frac{\text{sine}^2\, 10°}{\cdot 02} = \frac{\cdot 0301}{\cdot 02} = 1 \cdot 5 \text{ inches.}$$

Through the points of division, horizontal lines are drawn.

If now A B be produced until its length = 50 inches or 1·00 on the scale adopted, the sloping lines in the figure are drawn, by joining the points of division on the horizontal scale to the other end of the line so produced.

Fig. 10 is a small scale diagram to show this construction. It is not, however, actually necessary to produce A B for the full length. Suppose A C in figure be made 10″ long, so as to represent ·2 of an inch to scale, then, if C D be drawn horizontally, each division along C D will be ·8 or four-fifths of those on A E. Hence, the sloping lines can be drawn by setting off a scale on C D four-fifths as great as the scale of the plan, and joining corresponding points. In practice we need not go beyond the highest angle of elevation likely to be observed.

To use the scale, suppose the value of cs from the tacheometer be 476 feet, and the angle of elevation 8°. Enter the scale with the angle of elevation, and take the distance $a\,b$ (fig. 9) along the horizontal line for this angle, up to the sloping line drawn from 476 on the scale of feet. This distance $a\,b$ is simply

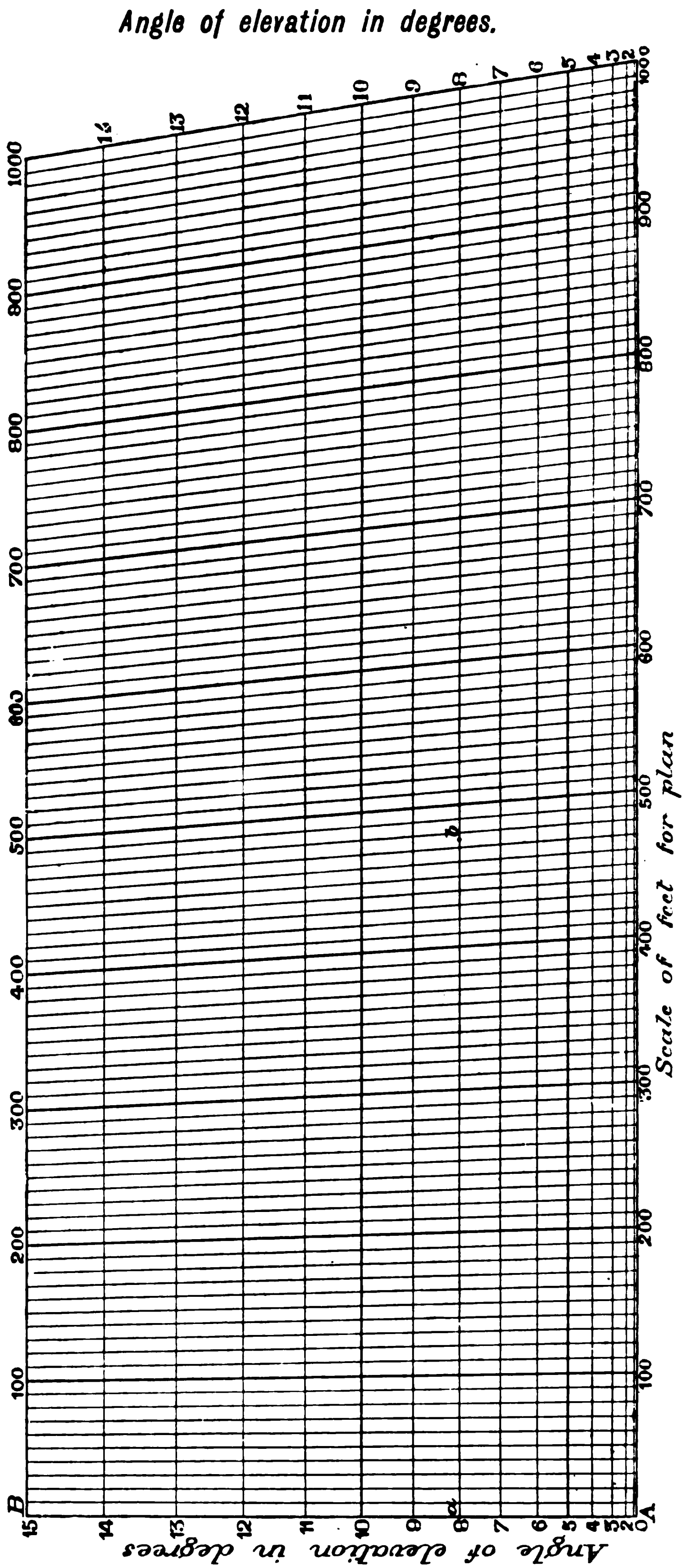

Angle of elevation in degrees.
Scale of feet for plan
Angle of elevation in degrees
A
B
a
b

Fig. 9.

taken off on the dividers and plotted direct on the plan. It is easy to prove from the construction that $a\,b = 476\,(1 - \sin^2 8°)$ $= c\,s \cos^2 a$.

Such a scale is particularly useful for plane-table work, and may be constructed on metal, if desired.

It is evidently useless for any other scale than the precise one for which it has been made.*

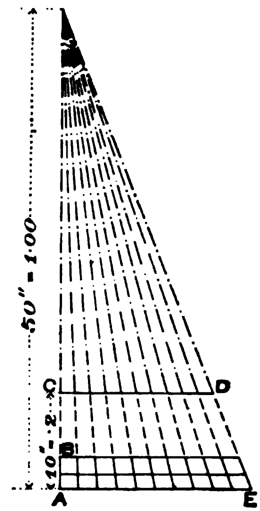


FIG. 10.

Difference of Level. The difference of level may be ascertained by reading the intersection of the centre wire, as well as the upper and lower wires.

Let $\Delta\,h =$ H O the difference of level between the axis of the telescope and the point intersected by the centre wire on the staff. Then

$$\Delta\,h = x \tan a = C\,s \cos a . \sin a - k \sin a$$

To find the actual difference of level between the instrument and object point,

mark $\Delta\,h\,(+)$ if the vertical angle is an 'elevation.'

„ $\Delta\,h\,(-)$ if the vertical angle is a 'depression.'

„ staff reading of centre wire $(-)$, height of instrument $(+)$, and sum algebraically.

The algebraical sum is the difference of level between the staff and instrumental points. If the 'negative' quantities are in the majority, the difference of level is a 'fall.' If the 'positive,' then it is a 'rise.'

Tacheometric Tables. To reduce tacheometric observations to the horizontal, rapidly, and to take out differences of level, an extended table giving the values of cosine² and cosine × sine for various apparent distances, is desirable. This table should resemble a traverse-table. Such tables have been constructed for both the quadragesimal and sexagesimal division of the circle. A specially constructed slide-rule is often used. In the absence of a proper table, an ordinary traverse-table will afford a ready means of reduction.

Enter the traverse-table with the apparent distance as a 'distance,' and the angle of depression or elevation as a 'bearing.' The difference of 'latitude' (cosine) so obtained, augmented if necessary by the constant focal correction, will be the *true* 'oblique distance.'

Entering again with the *true* 'oblique distance' as a 'distance,' and the angle of altitude, or depression, as a 'bearing,' the 'difference of latitude' (cos) will be the *true* 'horizontal distance,' and the 'departure' (sine) the difference of level, to which the proper sign must be applied according to the precept given above.

* When working with the Plane-table, it is not always desirable that the units on the staff should be decimals of a foot, or centimetres. For a Plane-table survey, on the scale $\frac{1}{2500}$, the staff was divided into units of ½ inch, numbered decimally. Each division of the staff, therefore, represented 50 inches on the ground. As 50 × 50 = 2500, one division, on the staff corresponded with $\frac{1}{50}$ of an inch on the paper, about the smallest division that the draughtsman can work to. The base-line, therefore, of the reduction diagram, on the preceding page, was divided into fifty parts to one inch.

Tacheometry with Ordinary Theodolite.

Tacheometric surveying may be effected, by means of any theodolite (which has a good and powerful telescope), merely by providing subtense wires in its diaphragm. When ruled on glass, in the manner described, subtense wires are always a useful addition to the 'level' or 'theodolite.'

As regards 'the level,' it is sometimes convenient to have graduations engraved on it giving *rises* or *falls* of one, two, three, etc., per thousand. The lines marked 5.5., fig. 11, act as ordinary subtense wires with a coefficient 100. The outer wires 10.10., give a coefficient of 50. When the telescope is horizontal, the minor graduations give rises or falls of one, two or three per thousand. In location of lines in rough ground this information is often useful. Suppose that a line of water-main is being surveyed. The hydraulic gradient is to be, say, 5 per thousand. Having reached a ridge, the surveyor finds that a valley and a second ridge have to be crossed. Levelling the instrument, the surveyor turns it until the graduation 5 cuts the opposite ridge, and thus ascertains the general direction in which he must proceed with the section.

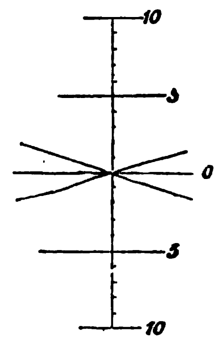


FIG. 11.

Merits of Tacheometric Methods of Surveying Discussed.

The telemetric telescope (with or without an anallatic lens) affords a most rapid method of measuring moderate distances with considerable accuracy, and this is especially the case when the telescope can be used in a 'horizontal position' as when 'levelling.' With a level fitted with a subtense eye-piece and a good compass, contours and trial sections can be run with ease and rapidity.

Reduction to the horizontal and computation of differences in Level.

The reduction to the 'horizontal' and the computation of 'differences of level' are somewhat laborious, especially in the absence of proper tables. When the 'horizontal distances' only are required, and are to be plotted direct, then their reduction may be effected by means of a special slide-rule (*vide* foot of p. 248, Part I.), or graphically as above described.

The accuracy of the measurements made with the telemetric telescope, depends largely on the power and optical precision of the instrument, on the distinct marking of the divisions on the staff, and to some extent also on the vision of the observer. The distance at which the single division of the staff can be read with certainty is limited. It requires a powerful and excellent telescope to bisect the hundredths of a foot at a distance of 400 feet.

Experience shows that, with the telemetric telescope, the probable error in measurement is not proportional to the distance measured. Up to a certain distance, dependent on the optical perfection of the instrument, the probable error is nearly constant. Beyond that distance, the errors become suddenly large and erratic.

For example, with a small and not very good theodolite telescope, it was found that measurement could be obtained to the nearest foot, up to 330 feet, but

beyond that distance, errors of two, three, and even four feet, occurred. With a more powerful and better telescope, the limit of accuracy appeared to occur at 800 feet.

Before commencing work with a new instrument, it is well to test it at various accurately measured distances, reading the staff, both with the telescope horizontal, and at the greatest angle of elevation and depression that circumstances will permit. The apparent distances should be reduced to the horizontal, tabulated, and compared with the measured distances. The surveyor will then ascertain the probable error in his telemetric measurement, due to all causes, and will know the limit of range, within which the desired degree of accuracy may be attained.

With a telescope of about 12″ focal length, of fair quality, distances may be obtained to about $\frac{1}{800}$th of the truth, up to about 400 to 600 feet. With a somewhat larger telescope, of very superior quality, the error should not exceed $\frac{1}{1000}$ th up to even 700 feet. Telemetric measurement, therefore, compares favourably with *all but the best* chainment.

The 'subtense-instrument with micrometer eye-piece' (so far as the optical side of the question is concerned), is on the same footing as the 'telemetric telescope.' A distinct object may be intersected with greater precision than can be attained by reading the graduations of a staff, whilst the distance between the wires can be measured with a high degree of precision, by means of a well made micrometer screw.

It is therefore probable that the 'subtense instrument' will give more accurate results than the 'telemetric telescope.' The Abyssinian survey shows that it is capable of high accuracy. Calculation is, however, required to obtain any distance, hence it is not so generally useful as the 'telemetric telescope.'

The determination of distances by 'vertical angles,' observed to the extremities of a fixed base, is equally dependent on the optical quality of the telescope with the methods hitherto discussed. Its accuracy depends, principally, upon the exact bisection of the marks on the staff, on the precision with which the 'vertical arc' can be read, and on the correctness of the divisions of the limb. With an ordinary theodolite, a single reading of a 'vertical angle' can scarcely be depended on to ± 15″, an angle which subtends 0′·00727 at 100 feet, or an error of ± 1 in 1376.

The following Table shows the *increasing* effect of this 'angular error' in proportion to the distance measured:—

TABLE OF 'MEASUREMENT ERRORS' DUE TO 'ANGULAR ERRORS' OF ± 15″.

Angular Equivalents.	Measurement Errors Resulting.
0′·00727 at 100 feet . . .	1 in 1376
0′·01454 „ 200 „	1 „ 670
0′·02171 „ 300 „	1 „ 461
0′·03008 „ 400 „	1 „ 332
0′·03635 „ 500 „	1 „ 275
0′·07271 „ 1000 „	1 „ 137

When the optical axis is 'horizontal' or nearly so, the effect of an 'angular error' is practically equal to an error in the length of the 'base,' equivalent to the distance subtended by the 'angular error' at the given distance.

Thus, at 200 feet the resultant error due to an angular error of 15″ is equivalent to an error in the length of the base of ·0145 of a foot. The base being 10 feet, the proportional error is ·00145 or $\frac{1}{690}$ of a foot. The preceding table gives the 'proportional error' for various distances for an error of ± 15″ in arc. It is therefore clear that this method, with an ordinary instrument, is not capable of high accuracy, except at short distances.

Both with the 'omnimeter' and the 'subtense-instrument' any given error in the measurement of the angle (or rather of its tangent in the case of the first-named instrument, and its chord in the second) produces a rate of error in the result, which increases in proportion to the distance measured, just as in the case of measurement by vertical angles. Thus, if it were found that at 1000 feet, an error of ± 1 per thousand is probable, the probable error at 2000 feet would be ± 2 per thousand, and so on.

The 'omnimeter' measures the 'tangents of angles' mechanically, with high precision. The optical quality of the telescope has little influence on the results. Theoretically, the 'omnimeter' should give a very high degree of accuracy, superior in fact to the other instruments described. In practice, this has not been found to be the case, owing to mechanical difficulties, connected with the micrometer screw, also, in that the distance between the microscope and the scale is variable, whilst the scale itself is not sufficiently finely and accurately graduated. To focus the horizontal scale, at different points of its length, the object-glass, or eye-piece of the microscope, or both, must move through a considerable space. A slight want of truth in the draw-tube of the microscope, will produce a material error in the measured tangent. In this instrument also, the error-fraction due to any given error in measuring the tangent, increases in direct proportion to the distance measured. The 'omnimeter' has this merit, that the reduction of the observation is extremely simple. By using a graduated staff and a fixed difference of tangent, horizontal distances may be read directly without any reduction. If all mechanical difficulties were overcome, the 'omnimeter' would be the most rapid and accurate of all the 'telemetric instruments' described.

CHAPTER IV.

BASE LINE MEASUREMENTS, AND METHODS OF CHAINMENT WHERE GREAT ACCURACY IS REQUIRED.

It is not possible, within the scope of this treatise, to give detailed descriptions of the methods hitherto adopted in base-measurements, but it is considered desirable to describe, generally, how such measurements have been conducted, as well as the instruments used, and the degree of *absolute* (not *relative*) accuracy obtainable.

A base measurement, being an absolutely necessary step in all triangulation (not in extension of previous surveys), a decision has to be made as to how it should be conducted, in order to meet the following requirements. Firstly, the degree of accuracy aimed at, which is dependent on the object with which the survey is undertaken, secondly, the time available, and thirdly, the funds to hand.

The implements available, arranged in order of rapidity of manipulation, are as follows—

1. Steel chains or bands.
2. Wooden, glass, or metal rods.
3. Compensation bars, such as Colby's.

The first Geodetic Survey in England was started in 1783, when it was decided to connect Paris with Greenwich, geodetically.

Hounslow Heath Base.

General Roy, appointed to carry out the work on the English side, commenced by measuring a base on Hounslow Heath in 1784.

The first measurement was made with a steel chain, 100 feet long. It was considered merely experimental, and gave for the length of the line, after correction for temperature, 27,408·22 feet.

Deal rods had been generally used in other countries, accordingly three thoroughly seasoned trussed deal rods, each 20 feet 3 inches long, were next used. They were each terminated in bell-metal tips, by the contact of which measurement was made, and were compared with a standard scale. During the work, it was noticed that the rods were much affected by the changes of humidity in the atmosphere (there is no record that they were oiled or varnished), so the measurement was considered unsatisfactory.

The result of this measurement gave 27,406·26 feet for the length of the line, reduced to the level of the lower extremity at the temperature of 63° F., being that of the standard brass scale, when the lengths of the deal rods were compared.

The base was next measured with glass tubes, 20 feet in length, the expansions of which were found by experiment. The temperature of each tube was obtained during the measurement, by two thermometers in contact with it.

This gave, when reduced to 62° F. and to the mean level of the sea, 27,404·0137 feet for the length of the line.

Reduction to M.S.L.

This reduction to M.S.L. (described later on) was then used for the first time in the history of geodesy.

The greatest care was taken during each of the measurements to secure correct alignment, to adjust differences of level, to estimate variations due to temperature, and to allow for all other possible sources of error. Where each day's work left off, a fine plumb-line was suspended to mark it off, the plummet vibrating in a brass cup, sunk in the ground and filled with water.

In 1791, when the work of the Ordnance Survey was resumed, it was decided to re-measure this base with a steel chain. Two chains of 100 feet in length were prepared by Ramsden. Each chain consisted of 40 links, half an inch square in section, with brass handles flat on the under side, a transverse line on each handle marking the length of the chain. One chain was used for measuring, the other as a standard.

The chain, stretched by a weight of 28 lbs., was laid out in a succession of deal coffers, carried on trestles, the temperature being always taken during the measuring of each chain length.

Five thermometers were laid close by the chain, in each position, and left till they showed nearly the same temperature, which took generally from 7 to 15 minutes.

The result of this measurement after reduction to 62° F. and M.S.L., was 27,404·24 feet, exceeding the length given by the glass tubes by ·21 feet, and falling short of that by the deal rods by 2·02 feet.

These measurements are given in terms of General Roy's brass scale.

Before further describing base-measurements, it is desirable to discuss the question as to what standard of length has been employed, not only in England but in other countries.

Standards of Length.

In measuring a base, the length has to be obtained in reference to some standard. The geodetic standards of length of different countries vary, both in form, and in the material of which they are composed. They are divided into two classes, standards '*à traits*' and standards '*à bouts*.'* In the first, the lines or dots defining the measure, are engraved on small discs of silver, platinum, or gold let into the bar. In the second, the bar generally has its extremities in the form of a small cylinder, presenting a circular disc, either plane or convex, of hard polished metal, or sometimes of agate, for the contact measurements.

Different Foreign Standards.

The unit of the length, in which by far the greater part of the geodetical measurements of Europe are expressed, is the *Toise of Peru*,† a measure '*à bouts*,' of which fortunately there exist two copies (compared with the original and certified by

* Expressed by distances from centre to centre of points or scratches, or by end-to-end measurement.

† See footnote to Appendix O.

Arago), one made for Struve in 1821, the second for Bessel in 1823. The standards of Belgium and Prussia are copies of the toise of Bessel. The Russian standard, which is two toises in length, is measured from the toise of Struve.

O.S. Standard. The standard of the Ordnance Survey is 10 feet in length (deduced from the standard yard), and is in section a rectangle of $1\frac{1}{2}$ inches in breadth by $2\frac{1}{2}$ in depth, supported on rollers at a quarter and three-quarters of its length. The ends of the bar are cut away to half its depth, so that the dots marking the measure of 10 feet are in the neutral axis.

Standard Yard. The standard yard of this country and its copies, are bars 1 inch square in section, of iron, steel, brass, or copper. The lines defining the yard, are in the axis of the bar. The length of the bar is that at 62° F., and is fixed by Act of Parliament which declares that 'the pendulum vibrating seconds of time in a vacuum, in the latitude of London, at the level of the sea, is 39·1393 inches of the standard, and that the yard shall be in the proportion of 36 to 39·1393 inches.'

Base Measuring Instruments. Mention has been made of the use of chains, glass and deal rods, in the measurement of the Hounslow Heath base. The steel chain has been found to give very good results, and has the great advantages of simplicity, portability, and cheapness, which will render it advisable to use it in countries where transport is difficult. But when extreme accuracy is required, in order to evade the temperature difficulty, different forms of measuring rods have been devised.

Borda's Rods. The apparatus used by the French, and constructed by Borda, consisted of two strips of metal in contact, forming a metallic thermometer, carried on a stout beam of wood. The lower strip was of platinum, two toises in length, and lying immediately on it was a strip of copper, about 6 inches shorter, fastened to it at one end only, so that it was free to move along the platinum, as its relative expansion required. A graduated scale at the free end of the copper, and a corresponding vernier on the platinum, indicated the varying relative lengths of the copper, whence were inferred the temperature and length of the platinum strip. At the free end of the latter, uncovered by copper, was a graduated slider, moving in a groove, which was used to measure the interval between two successive platinum strips.

Struve's Bars. The Russian astronomer, F. W. Struve, used a wrought-iron bar, two toises in length, one end terminating in a small steel cylinder, its end being slightly convex and highly polished. The other end carried a contact lever of steel, the lower arm of which terminated in a polished hemisphere, and the upper arm traversed a graduated arc, also rigidly connected with the bar. The length of the bar was known to whatever division of the arc the index line at the end of the lever pointed. In measuring, the bars were brought into contact, which was maintained by a spring action on the lever. Two thermometers, whose bulbs were let into the body of the bar, showed its temperature.

The probable errors of the seven bases measured with these bars range from $\pm\ 0{\cdot}73\,\mu$ to $\pm\ 0{\cdot}914\,\mu$, where

$$\mu = \text{a millionth part of the length measured.}$$

Bessel's System.

The Germans used Bessel's system, in which the platinum and copper of Borda were replaced by iron and zinc, and the intervals were measured with a glass wedge. The upper or zinc rod terminated at either end in a horizontal knife-edge. The rods were supported on seven pairs of rollers, carried by a bar of iron.

The probable error of Bessel's base was found to be $\pm\ 2 \cdot 2\ \mu$.

This apparatus was used in the Belgian bases, measured (1852–53) by General Nerenburger with every precaution, particularly as to fixing the end of each day's work most minutely.

The mean errors were then computed to be, for the Beverloo base, 2300 metres in length, $\pm\ 0 \cdot 59\ \mu$, and for the Ostend, 2488 metres in length, $\pm\ 0 \cdot 45\ \mu$.

U.S.C. Survey Apparatus.

The United States Coast Survey Base Apparatus, devised by Professor Bache in 1845, combined the principle of Borda's measuring rods, the compensation tongue of Colby's, and the contact lever of Struve's. The cross-sections of the bars were so arranged, that while they had equal absorbing surfaces, their masses were inversely as their specific heats, allowance being made for their difference of conducting power. The components were placed edgeways, the iron above and the brass below, firmly united together at one end. The brass bar, which had the largest cross-section, was carried on rollers mounted in suspending stirrups, and the iron bar rested on small rollers, which were fastened to it, and ran on the brass bar. Supporting-screws, through the sides of the stirrups, retained the bars in place. The connection between the free ends of the component bars, was the lever of compensation, which was pivoted to the lower bar. A knife-edge, on the inner side of this lever, abutted against a steel plane, on the end of the upper or iron bar. At its upper end, this lever terminated in a knife-edge, facing outwards, in a position corresponding to the compensation point in Colby's bars. The knife-edge pressed against a collar on a sliding rod, moving in a frame affixed to the iron bar above. The sliding rod was drawn backwards by a spiral spring, through which it passed, and kept the lower knife-edge of the lever pressed, with constant pressure, against the iron bar. The sliding rod terminated in an agate plane, for contact. A vernier, attached to this end of the bar, gave their difference of length, as a check on the work.

At the other end, where the bars were united, there was a corresponding sliding rod terminating outwardly in a blunt horizontal knife-edge. The inner end abutted against a contact lever pivoted below. This lever when pressed by the sliding rod came in contact with the short tail of a level, mounted on trunnions and not balanced. For a certain position of the sliding rod the bubble came to the centre, and this position gave the true length of the measuring bar. This was an exceedingly delicate mode of measuring.

At this end of the apparatus, there was also a sector for indicating the inclination of the bar in measuring, and it is to the arm of this sector that the contact-lever and level were attached.

As in Colby's apparatus, however, the compensation cannot always be absolutely relied on. The length of the bar depended on whether the temperature was

rising or falling, and a length had to be assigned, from actual comparisons in each of these conditions.

Eight or more bases have been measured with these bars, which offered considerable facility for rapid work, as much as a mile in one day having been completed with them.

One of the last bases, that of Atalanta in Georgia, was measured twice in winter and once in summer, the temperatures extending from 18° F to 107° F., by which means an extreme test of the performance of the bars was afforded.

The probable error of this last base was $\pm 1 \cdot 76\,\mu$, whilst those of the seven previously measured varied from $\pm 1 \cdot 8\,\mu$ to $2 \cdot 4\,\mu$.

Porro's System.

A wholly different system was that of Porro. Only one measuring bar was used, composed of two cylindrical rods of steel and copper laid side by side, firmly united at their common centre, and free to expand outwards. With this bar, the successive equal intervals between microscopes, arranged in the line of the base, were measured.

A modification of Porro's apparatus, as made by Colonel Hassard, was used for three bases in Algiers, giving probable errors of $\pm 1 \cdot 0\,\mu$ in each case.

Colby's Compensation Bars.

On the Ordnance Survey, the Compensation Apparatus, invented by Major General Colby, already alluded to, was used.

The upper bar was of iron, the lower of brass, 10 feet in length, firmly connected at their centres. At either extremity was a metal tongue, about 6 inches long, pivoted to both bars, so as to be perfectly firm and immovable, while yet not impeding the expansion of the bars. A silver pin let into the end of each tongue, carried a microscopic dot, marked $c\,c'$ (*vide* fig. 12). The letters $a\,b$, $a'\,b'$, refer to the axes of the pivots shown by dots.

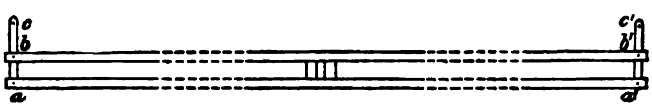


FIG. 12.

Now if α, β, be the rates of expansion of the brass and iron bars $a\,a'$, $b\,b'$ respectively, by construction,

$$a\,c : b\,c = \alpha : \beta = a'\,c' : b'\,c'$$

Now the centres of the bars being fixed, let an increase of temperature imparted to the bars cause a to move off the small distance $\alpha\,i$, while b is carried in the same direction the amount $\beta\,i$. It is clear that the movement of c is zero, that is, the distance of the dots $c\,c'$ remains 10 feet.

In order to ensure the proper action of this mechanism, the radiation and absorption of heat by the bars must be equal. This was effected by clouding and varnishing the surfaces until, by experiment, the rates of heating and cooling were found to be the same.

This compound bar was placed in a deal box (resting immediately upon two brass rollers in the bottom of it), and kept from moving by means of a pin, fixed in the bottom of the box.

The complete set contained six bars. Each box, when in use, was supported at a quarter and three-quarters of its length, by strong brass tripods, having rollers on their upper surfaces.

The interval between two adjacent bars was exactly 6 inches, and was measured by a 'compensation microscope.'

The end of each series of six bars in the measurement was transferred to the ground by means of a 'point carrier,' which was a massive triangular plate of cast-iron, having attached to its surface, or at a height above it that may be varied as required, an adjustable horizontal disc, with a fine point engraved on it. This point was adjusted to bisection, in the focus of the advanced telescopic microscope.

The Lough Foyle and Salisbury Plain bases, and ten bases in India, have been measured on this system.

Unfortunately, these bars have not given unqualified satisfaction, especially in India, where they were tested on a portion of the Cape Comorin base.

The probable error in the measurement of a base line with this apparatus is about $\pm 1 \cdot 5\,\mu$.

Lough Foyle Base.

On the Ordnance Survey, Colby's bars were first used in the measurement of the Lough Foyle base in 1827–8.

Salisbury Plain Base Re-measured.

In 1849 it was determined to re-measure the Salisbury Plain base with the same apparatus, it having before been measured with a chain in 1794.

Salisbury Plain is very well adapted to the measurement of a base, and a longer line might have been selected on it, but it is certain that little if any, advantage is gained by the measurement of a base of more than 6 or 7 miles, provided it be surrounded with very careful triangulation.

Comparison of Bars with O. S. Standard.

The bars were most carefully compared with the Ordnance Survey standard. The standard was, in every case, first brought under the microscope, then the six compensation-bars in succession, and then the standard again. The temperatures of the bars, as well as the readings of the micrometer, were registered. The standard bar has the bulbs of two mercurial thermometers let into it, and the interstice being filled with oil, the effect of the change in temperature of the air is avoided. Some 65 sets of comparisons were taken during the measurement of the base.

Result.

After all the necessary corrections, &c., were made, the length of the line stood thus—

Measured with compensation bars		34,840·8579
" " intermediate compensation microscopes		1,741·9234
" back with beam compass (from nearest multiple of bars)	—	4·2938
Reduction to level of sea	—	0·6294
Length of base line in feet of Ordnance Survey standard		36,577·8581

Broken Base.

It is always best to measure a base in a straight line, between the extremities, but on account of the difficulty in finding a

suitable place of sufficient length, it is sometimes necessary to measure a 'broken base,' as was done in the bases of Melun and Perpignan.

In fig. 13, suppose A C B to be the broken base, of which the lengths of A C and C B are known, also the angle A C B, which may be supposed to be nearly 180°.

Let B C I $= \theta$, a very small angle
A C B $= 180° - \theta$
C B $= a$
A C $= b$
A B $= c$, the length of base required

Then

$$c^2 = a^2 + b^2 + 2ab\cos\theta$$

and as θ is very small, $\cos\theta = 1 - \frac{\theta^2}{2}$

$$\therefore \quad c^2 = a^2 + b^2 + 2ab\left(1 - \frac{\theta^2}{2}\right)$$

$$= a^2 + b^2 + 2ab - ab\theta^2$$
$$= (a+b)^2 - ab\theta^2$$

Whence $c = (a+b)\left[1 - \frac{ab\theta^2}{(a+b)^2}\right]^{\frac{1}{2}}$

Expanding this series, and letting θ denote the number of minutes in the angle,

$$c = a + b - \frac{ab\theta^2 \sin^2 1'}{2(a+b)} \text{ (very nearly)}$$

Again, to find the angle B A C or A

$$\frac{\text{Sin A}}{\text{Sin }\theta} = \frac{a}{c} \text{ and } \sin\theta = \theta - \frac{\theta^3}{6}$$

$$\therefore \quad \sin A = \frac{a\theta}{c}\left(1 - \frac{\theta^2}{6}\right)$$

Substituting above value for c,

$$\sin A = \frac{a\theta}{(a+b)\left[1 - \frac{ab\theta^2}{(a+b)^2}\right]^{\frac{1}{2}}} \times \left(1 - \frac{\theta^2}{6}\right)$$

$$= \frac{a\theta}{a+b}\left[1 + \frac{ab\theta^2}{2(a+b)^2} - \frac{\theta^2}{6} - \frac{ab\theta^4}{12(a+b)^2}\right]$$

Neglecting θ^4, etc.

$$= \frac{a\theta}{a+b}\left[1 + \frac{3ab - a^2 - 2ab - b^2}{6(a+b)^2}\theta^2\right]$$

$$\therefore \quad \sin A = \frac{a\theta}{a+b}\left[1 + \frac{ab - a^2 - b^2}{6(a+b)^2}\theta^2\right]$$

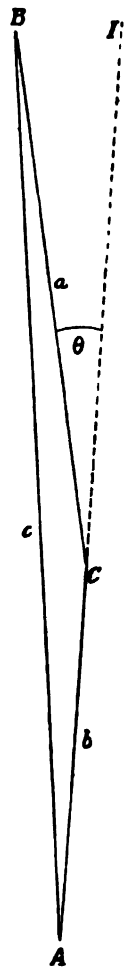


FIG. 13.

Now $$A = \sin A + \frac{\sin^3 A}{6}$$

Substituting this value for sin A, and putting the small angles A and θ equal to A sin 1′ and θ sin 1′ to express them in minutes, we have

$$A = \frac{a\,\theta \sin 1'}{a+b} + \frac{a\,b\,(a-b)\,\theta^3 \sin^2 1'}{6\,(a+b)^3}$$

Prolonging a Base.

It is sometimes desirable to increase the length of a measured base, over ground which does not lend itself conveniently to direct measurement, owing to some kind of obstacle, such as a river, ravine, or broken ground, intervening. In such a case the completed measurement may be prolonged to the desired position of the base terminal on the following principles:—

In fig. 14, suppose A C to be a base line of which A B has been measured, E and D having been fixed during the measurement. Select stations F and G so

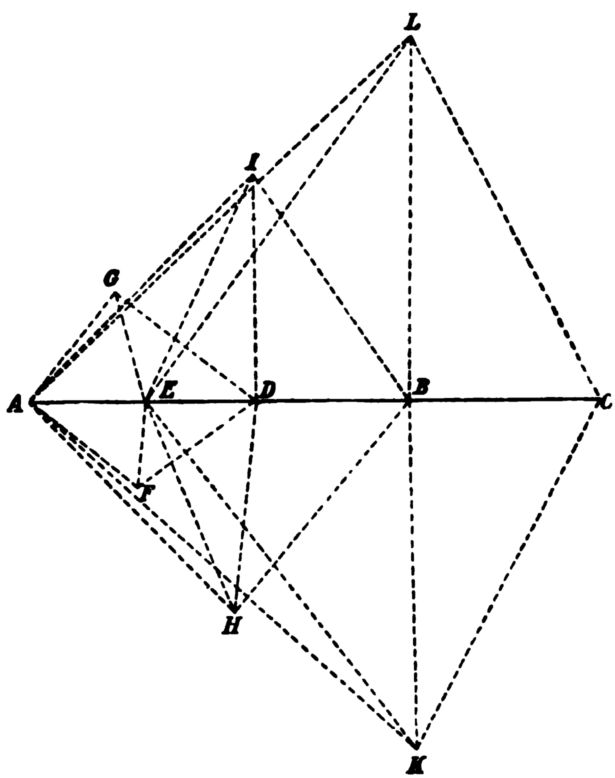


FIG. 14.

that the angles at E may be nearly right angles and the points themselves nearly equidistant from the line, and about equal to A E. Similar conditions determine the position of H, I, K and L. At A and all the other points along the line and

on either side, all the points visible are observed with a theodolite. From A E and the observed angles, E F and E G are determined, from each of which in the triangles D E F, D E G, the side E D is obtained, the distances thus found forming two checks on the measured length. D H and D I are in like manner calculated from A D and E D as bases, and each of these again furnish data for the determination of D B. Lastly, B L and B K are found from A B and also from E B.

From the mean results of these, B C is obtained, thus giving an addition to the measured portion of the base. This latter operation is called the 'prolongation of a base,' and is often convenient, or necessary, in order to complete, or extend a base.

The measured portion of the Lough Foyle base was about 8 miles, and two miles were afterwards added by this method.

By subsidiary Bases. The prolongation of a base may also be performed by the help of 'subsidiary bases,' measured for that purpose. Suppose for instance in the last figure that C E had been measured, then E F and E G would be measured as subsidiary bases, and the length E A calculated by using them and the angles observed from E, F, G, and A.

Base of Verification. In all large surveys a 'base of verification' should be measured, that is to say, a base on a side of a triangle at some remote part of the survey. A comparison between the lengths of this base, as found by measurement and calculation, forms a check on the accuracy of the calculation of the triangles. This is the severest test to which a geodetic operation can be subjected.

It may be mentioned that it was determined to use the Salisbury Plain base, (the detail of the measurement of which has been given), as a base of verification, taking the measured length of the Lough Foyle base as correct. The distance between the two bases is 360 miles, and the difference between the measured and calculated lengths was less than 5 inches.

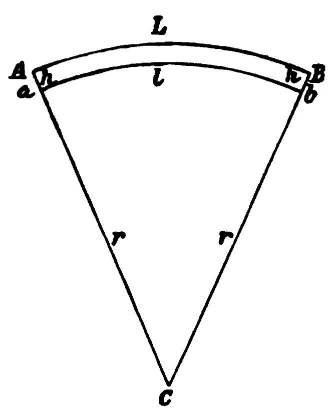


Fig. 15.

Reduction to Sea-level. In order that there may be a common datum, all observations and measurements are reduced to their values at mean sea level.*

In the case of a base line (see fig. 15)

Let $AB = L =$ the length of the base line reduced to the horizontal.

$ab = l =$ its value at the sea level.

$Aa = Bb = h =$ the mean height above the sea level.

$Ca = Cb = r =$ the radius of the earth.

Then $CA : Ca :: AB : ab$

* The Geodetic Surface is defined in the article on Determination of Heights and on Levelling in the First Part of this treatise.

$$\therefore \qquad l = \frac{rL}{r+h} = L\left(\frac{1}{1+\frac{h}{r}}\right) = L\left(1+\frac{h}{r}\right)^{-1}$$

$$= L\left(1 - \frac{h}{r} + \frac{h^2}{r^2} - \ldots\right) = L\left(1 - \frac{h}{r}\right) = L - \frac{Lh}{r}$$

neglecting higher terms of $\frac{h}{r}$ (*Vide* Part I.)

Rapid Base Measurement.

A considerable amount of accuracy, with great rapidity in the measurement of a base, may be obtained by the use of a fine steel wire of say 2000 feet in length, stretched between two points and having a known versed sine of say 25 feet. It is only necessary in this case to measure a length of 2000 feet with the greatest possible accuracy, to suspend the wire between these points, and having calibrated it, to extend it successively between fixed stations 2000 feet apart (on which to support the ends), keeping careful records of the temperature. This process, however, could only be carried on in still weather.

Estimates of Accuracy.

All estimates of the accuracy of base measurements must be accepted with reserve, for they must necessarily be relative, and comparison is frequently made between two or more measurements made with the same instruments, and the same staff, over the same ground, the result being, not the approximation to absolute accuracy, but the *difference* between certain measurements which may, or may not be accurate, either as regards their units or the aggregation of these units.

In Gore's 'Elements of Geodesy,' New York, 1886, the accuracy obtainable in base line measurements is stated to be as high as 1 in 6,000,000.

Methods of Chainment where Great Accuracy is Required.

Of recent years, it has become the practice, especially in the United States and the Colonies, to substitute the steel band for the link chain, with the result that greater rapidity and accuracy in making lineal measurements are attained.

Although the link chain is still in common use in England, the objections to it are numerous. Being built up of many parts, there is a constant tendency to increase in length, and this must be as constantly counteracted by frequent reference to a standard. The weight of the chain is considerable, it picks up earth, grass and sticks, in being dragged along the ground, and is particularly liable to catch in passing through shrubs and fences. At the best, as already pointed out under 'Chain Surveying,' in Part I., the accuracy of measurements made with it is not great.

Some thirty or forty years ago an attempt was made to substitute for the link chain, a steel band about $\frac{3}{4}$ inch wide and less than $\frac{1}{32}$ inch thick. The band did not stretch, and was in this respect an improvement, but unless it was carefully cleaned, the link marks were soon obliterated. The steel being hard was very

liable to kink, and was frequently broken in stretching, when it could only be mended by brazing or riveting, and, moreover, the sharp edges cut the hands of the chainmen, who objected to its use.

More recently a steel band, varying from $0''\cdot1225$ by $0''\cdot005$ to $0''\cdot25$ by $0''\cdot02$ (the latter only weighing $\frac{3}{4}$ oz. per yard), with the corners slightly rounded, has been adopted, and is decidedly preferable to the link chain, for the following reasons. It is light, does not pick up dirt or grass, and does not stretch, whilst it is handier than the old steel band and less cumbersome, not cutting the hands, and not being liable to kink.

The lighter section has also the advantage of offering less surface for the wind to act on, or for dew to deposit on. The ordinary length of these bands varies from 66 feet to 100 feet.

The light steel band described, is frequently marked, at every link, by a strip of brass attached to the band, and at every ten links by a broader strip of brass, the number of links from either end of the band being marked on the brass strips with centre punch indentations. In some bands, used for special purposes, the lengths are engraved on the tape.

In this respect the band is inferior to the chain, as the links are more difficult to read, and require close inspection.

Although the steel band does not itself stretch, the handles do, and for accurate work the handles should be outside the chain proper, and should not be measured. The narrow band is still liable to kink, and it would be improved if a material could be found, sufficiently soft not to kink, sufficiently tenacious to maintain its length, and sufficiently stiff to prevent twisting on itself.

With a chain or band as described above, measurements may be made to a high degree of accuracy, limited by the standardising of the band itself, the amount of accuracy with which variations of temperature *of the metal* can be recorded, and the degree of care employed in marking down the end measurements.

A method of using the measuring band may be described as follows.

The back or trailing end of the band is attached to a hook, which can be traversed through a small distance (in the direction of measurement), by means of a screw, passing through a lug on a flat plate, which rests on the earth, and is held in position by the foot or knee, or by the use of a lever in the form of a rod, hinged to the above-mentioned plate, and held in position and tightened or slackened by means of a guy-line attached to the rod.

The front or leading end of the chain is provided with a similar apparatus, but between this end of the band and the hook, a spring balance is introduced.

When the chain is in position (*the two ends being level*), sufficient strain is put on the front end by means of the screw or lever, to show from 10 lbs. to $25\frac{1}{2}$ lbs. on the spring balance, according to the length and weight of the band used.

The screw or lever at the back end is then slackened, or tightened, until the mark on the chain is exactly over the commencement mark on a peg or stone below it.

Finally, the strain on the front end is reduced, or increased, until the spring balance records the required strain.

The position of the end of the chain may be transferred to a peg or other mark on the ground by means of a plumb-line, or preferably by the use of a theodolite.

A good method of marking the leading end of the tape is by means of a vernier, fixed on the head of a stake, and raised by a screw under the tape. The graduation extends 9 mm. each way, which, being divided into ten parts, reads $\frac{1}{10}$ mm. The screw is provided with a hook, in the centre of its head, to which a plumb-bob may be attached.

The band being attached only at the two ends, and strained to the extent of a given number of pounds, will be slightly curved, and therefore the distance between the two marks will be slightly less than a chain. A calculation must be made of the amount of this defect, and this must be added to the length read on the band.

On inclined ground, it may not be convenient or possible to arrange for the two ends of the chain being level. In this case, the inclination of the band must be measured, and the necessary reduction to the horizontal length calculated.

When the two ends of the band are at different levels, the shape of the catenary will be somewhat altered, but the variation will be so small that, except where very great accuracy is aimed at, it may be ignored.

The apparatus for holding and moving the ends of the band may be varied in many ways, and the strain on the spring balance may also be varied if allowance be made for the same.

On suitable ground, bands five chains in length may be used with advantage, the defect in length due to the curvature of the chain being added to the five chain length itself.

When a band 66 feet in length is used, intermediate supports are unnecessary, nor are they required when a band of greater length is employed, except to reduce the effect of wind-pressure.

The true length of a band or tape may be determined in the following manner. On a level piece of ground, supporting stakes are driven 30 feet apart, the heads being ranged in line, and levelled with a theodolite. The tape is then secured in position, and the tension applied. Its length can then be measured with a standard bar, the correction to the same for temperature being applied, or, preferentially, a bar packed in ice may be used, to obviate the necessity for this. The temperature of the tape is constantly observed during the operation, which is repeated from 50 to 100 times, the differences being averaged by the system of least squares. By this means the probable error in the length of a band can be reduced to $\frac{1}{500000}$ of the total length.

Method of procedure for important Measurements.

When important measurements have to be made, 'support' and 'marking' stakes are arranged and adjusted beforehand, at intervals of from 10 to 20 yards or metres. When measuring, the rear extremity of the band is first adjusted (by means of a slow motion screw), so as to bring the zero of the band exactly over the initial mark. The band is then stretched till the required tension is indicated on the spring balance, when the leading end is marked and the exact distance noted. The readings of the thermometers are recorded, and the band advanced. About two kilometres per hour can be measured in this way.

To obviate the tendency of the temperature of the band to lag behind that of the air, measurements should first be made with a *rising* or falling temperature and secondly *falling* or rising temperature, respectively. The measurements can be repeated any number of times, in opposite directions, to acquire the desired degree of accuracy, by taking a mean of the same, and in this operation the differences should not exceed $\frac{1}{180000}$ of the whole length.

Reduction to the Horizontal. If the ground be on a slope, the inclined measured length must be reduced to the horizontal.

If L = the inclined measured length, in feet

L_1 = the horizontal length, in feet

h = the difference of level between the ends, in feet

Then,

$$L_1 = \sqrt{L^2 - h^2}$$

Correction for Temperature. Since the band has been compared with a standard at a given temperature, say 62° F., and its length increases in proportion to a rise in temperature, and *vice versâ*, it is always necessary to ascertain its length, at the temperature at the time of measurement. The *air* temperature must not be used in this case, but the nearest possible approximation to the temperature of the metal of the band itself.

If A = length of band, as compared with a standard, at a given temperature

x = rate of expansion of the band

t = temperature of band at time of measurement in degrees F.

Then,

$$L = A + x\,t$$

On the survey of the City of Sydney, when a 66-foot steel $\frac{1}{4}''$-band was used, it was found that the rate of expansion was equal to ·005 of an inch, per degree F., and this is in close agreement with the simple rule 'allow $\frac{1}{32}$ of an inch per degree F. in every 400 feet of length.'

Calculations necessary when using a supported Band. (1) In order to calculate the change in length due to a change in the tension applied,

Let L = normal length of the band, or *rectilineal* distance between the end marks under the standard tension.

ΔL = change in length, due to a change Δt in tension

w = unit weight of band

t = tension applied

$a = \dfrac{w}{t}$

n = number of sections between the supports

l = distance between the supports

$\Sigma l = n\,l$, approximately

μ = reciprocal of the product of the modulus of elasticity of the band, multiplied by its sectional area.

Then, for a change Δt in tension

$$\Delta L = n l \mu \Delta t + \frac{1}{12} a^2 n l^3 \frac{\Delta t}{t} \quad . \quad . \quad . \quad . \qquad (a)$$

Example.—

$\Delta t = 1$ ounce, with $t = 408$ ozs.

Let $n = 10$

$l = 10$ metres

$w = 22 \cdot 32$ grammes per metre

$t = 25 \cdot 5$ lbs. $= 408$ ounces $= 11{,}566\frac{2}{3}$ grammes

$a^2 = 372 \times 10^{-8}$

$\mu = 16 \times 10^{-9}$ for gramme as unit *

$= 450 \times 10^{-9}$ for ounce as unit.

Then

$$n l \mu \Delta t = 10 \times 10 \text{ m.} \times 450 \times 10^{-9} \quad = 0 \cdot 045 \text{ mm.}$$

and

$$\frac{1}{12} a^2 n l^3 \frac{\Delta t}{t} = \frac{1}{12} \times 372 \times 10^{-8} \times 10 \times 10^3 = 0 \cdot 0076 \text{ mm.}$$

Hence,

$$\Delta L_1 = 0 \cdot 0526 \text{ mm.}$$

(2) Suppose a given band to be supported, firstly, at n_1 sections of length l_1, and secondly, at n_2 sections of length l_2, and assuming $n_2 > n_1$, then, the difference in rectilineal distance between the end graduations of the band, in the two cases, Δl_3 say, will be expressed by the equation

$$\Delta l_3 = \Sigma (l_2 - l_1) = \frac{1}{24} a^2 (n_1 l_1^3 - n_2 l_2^3) \quad . \quad . \quad . \qquad (b)$$

Example.—

For a 100 metre band, supported at every 20 m., and therefore $n = 5$,

$$\Delta L_2 = \frac{1}{24} \times 372 \times 10^{-8} (5 \times 20^3 - 10 \times 10^3)$$
$$= 4 \cdot 65 \text{ mm.}$$

If one support be omitted, we have only to make $n_2 = 2, n_1 = 1$ and $l_1 = 2 l_2$ in equation (b).

Under the tension of $25 \cdot 5$ lbs., the band would be shortened by $0 \cdot 93$ mm., when $n_2 = 10$ and $l_2 = 10$ m.

Similarly, the omission of m consecutive supports shortens the distance measured by the band, by $\frac{m}{24} (m+1)(m+2) a^2 l^3$, where l is the length of section when no supports are omitted.

* Since the cross-section of the band under discussion is $6 \cdot 33$ mm. $\times$ $0 \cdot 47$ mm. or $00 \cdot 298$ square cm., the value of μ corresponds to a modulus of $2 \cdot 1 \times 10^6$ kilos. per square cm., or 30×10^6 lbs. per square in.

(3) If supported throughout its length in a trough, then,

$$\Delta L_3 = L_0 - \Sigma l = \tfrac{1}{24} a^2 n l^3 \quad . \quad . \quad . \quad (c)$$

(4) When the unit weight changes, owing to wear and tear, or the deposit of dew,

$$\Delta L_4 = - \frac{1}{12} n l^3 a^2 \frac{\Delta w}{w} \quad . \quad . \quad . \quad . \quad (d)$$

For further information on this subject the student is referred to the 'Transactions of the American Society of Civil Engineers,' vol. xxx., page 81.

CHAPTER V.

RAILWAY CURVES.

Railway Curves in general use. CIRCULAR curves are in most general use, though transition curves, which are considered on page 6, are coming into favour.

Designation. Curves are designated from their character as: Simple (Fig. 16), Compound (Fig. 25), or Reverse (serpentine) curves (Fig. 27).

Simple Curve.—A Simple curve is of the same radius throughout, from straight tó straight.

Compound Curve.—A Compound curve is composed of two or more simple curves of different radii, having a common tangent at the point of meeting of each pair of curves.

Reverse Curve.—A Reverse curve is composed of two simple curves of contrary flexure, which either have a common tangent at their point of meeting, or, as is usual, a short straight joining them and tangential to both.

The use of the common tangent is to be deprecated, except in crossover roads, as it necessitates the reduction of the super-elevation of the outer rail from its maximum to nil by the time the end of the curve is reached. At least 132 feet (2 chains) of straight line should be introduced between two reversed curves and where the curves are of small radii, the length of the straight line should be increased to 198 feet (3 chains) or 200 feet (2 chains of 100 feet) so that the maximum super-elevation of 6 inches may be taken out in this length, half in the curve and half in the straight, the super-elevation for the reverse curves being put in in a similar length, which can be done without straining the machinery of the locomotive. Transition curves may with advantage be used instead of the straight (see paragraph on transition curves).

Nomenclature. In England, in accordance with the Standing Orders of the Houses of Parliament, railway curves are defined by the lengths of their radii, in chains of 66 feet or 100 links, as curves of 10 chains, 20 chains, etc., radii. In some other countries a similar nomenclature is followed, the chain used being either 20 metres (65·618 feet), or 100 feet in length.

Degree Curves. In America and in much recent practice they are termed 1° curves, 2° curves, etc., the number of degrees indicating the angle at the centre subtended by an arc of 100 feet. For instance a circle described with a radius of 5730 feet has a circumference of approximately 36,000 feet and as there are 360° in a circle, every 100 feet of arc subtends an angle at the centre of 1°.

Many theodolites for use abroad are divided into 400° in which case the radius of a 1° curve becomes 6366 feet or 96·45 chains of 66 feet.

In open country, or where the exact position of the curve is of small importance, the degree nomenclature is not disadvantageous, but, where the position of the curve is limited, it may be to inches, as in the case of a line passing over valuable property or in a tunnel, the method of calculation by radii should be adopted.

Length of Chord.

In setting out curves in practice the lengths of the chord and arc are always assumed to be equal, but for this to be true within the necessary limit of accuracy, the length of chord must never exceed $\frac{1}{20}$th of the radius. Thus curves of less than 10 chains radius should be set out with $\frac{1}{2}$ chain chords, curves of 10 to 20 chains with chain chords and curves of 20 to 40 chains with 1 chain chords, while chords of 2 chains may be used for curves of over 40 chains radius.

Chains.

The chains most used in practice are the Gunter's chain of 66 feet, the 100 feet chain and the 20 metre chain. Tables applicable to each length are available, but where none are handy, any required tangential angle may be worked out (see paragraph on tangential angles).

The various Methods of laying out Railway Curves.

The various methods of laying out circular curves may be conveniently classed as follows:—

1. Where a theodolite, capable of laying out horizontal angles with the greatest accuracy, is used.

2. Where an instrument, other than a theodolite (i.e. sextant, prismatic compass, etc.), capable of laying out horizontal angles, but with less accuracy, is employed.

3. Where chain measurements are used, the fundamental angles being laid out by a horizontal measuring instrument or laid down on a plan and measured in therefrom.

1. Where a Theodolite is used.

Under the first head classification may be made as follows:—

(*a*) By the use of Tangential Angles, with one instrument.

(*b*) By the use of Tangential Angles and two instruments.

(*c*) By the use of Deflection Angles, one instrument being employed.

It must be borne in mind that, before any curve is actually pegged out on the ground, the straights approaching it are laid out and cannot, as a rule, be altered in direction without interfering with the junction to some other curve already set out. Under these circumstances it is obvious that, though they may be known approximately, neither the exact point of intersection of the tangents, the lengths of the tangent lines, the positions of the tangent points on the straights nor the radius of the curve has been determined.

Speaking generally, the radius of the curve as laid down on the plan is adapted, the tangent points being altered to suit. There are, however, cases, in populous districts, in very sidelong ground and in tunnel, where the apex of the curve must be adhered to, in which cases it will probably be preferable to alter the radius slightly and to maintain the tangent points as set out.

To lay out a curve without determination of the angle of intersection of the

tangents, the position of the tangent points and the radius and length of the curve to be used is waste of time, as the work will almost certainly be a misfit, but, if the several observations referred to be taken with exactitude, and ordinary care be used in laying out the curve, the junctions with the straights at the two ends should be perfect.

Method of Tangential Angles with one Instrument.

Problem.—Given two straight lines A P, P B (Fig. 16) to join them by means of a circular arc. The surveyor is supposed to have laid down roughly on the ground the starting and finishing, or tangent, points A and B of the curve, and, having laid these down on a plan, to have estimated a radius suitable for the curve.

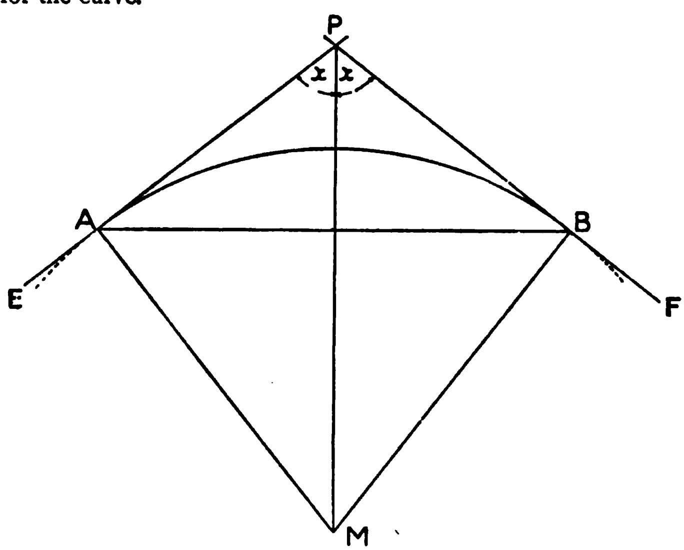


FIG. 16.

In order to do this the triangle A P B must be solved, and it is to the methods of determining the value of this triangle that the following lines are devoted.

The "Apex Angle" A P B is the supplement of the "Central Angle" A M B, the tangent lines A P, P B being equal.

In the completed figure (Fig. 16).

$$\text{Angle } PAB = 180^\circ - EAB.$$
$$\text{Angle } PBA = 180^\circ - FBA.$$
$$\text{Angle } APB = 180^\circ - (PAB + PBA).$$

and

$$\text{Angle } APM = \text{Angle } BPM = \tfrac{1}{2} \text{ Angle } APB.$$

To determine the value of the triangle A P B, let the angles E A B and F B A be observed with the theodolite, then the "Apex Angle" A P B = (E A B + F B A) − 180°. If the tangent points have been laid out correctly, angle E A B should equal angle F B A, but whether this is so or not the value (E A B + F B A) − 180° will give the correct result for the "Apex Angle" A P B.

Then if P is visible and accessible from A or B, let the distance A P or B P be chained, or, if B is visible and accessible from A, A B may be chained, preference being given to the shorter or easier line.

Should P be invisible from A or B or from both, but accessible from one of them, the tangent lines, A P, B P may still be ranged in with the theodolite to their point of intersection, and A P or B P may be chained.

If P is inaccessible from both A and B and the point B from the point A, some other points on the tangent lines, or the approaching straights, must be sought, which are both visible and accessible from one another, as K and L or N and Q (Fig. 17). In such a case the angles A K L and K L B, or N A B and A B Q, must be observed and the base K L or N Q, measured.

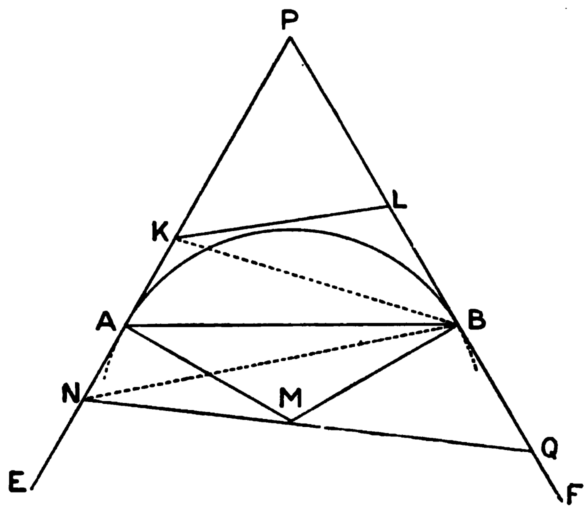


FIG. 17.

In the event of none of the methods described above being applicable, there are still many other ways in which the desired solution of the triangles A P B, or of a substituted triangle in which A P B is still the apex angle may be arrived at.

For example, the inaccessible portion of the line A P or P B or of both, may be bridged by triangulation. Again, let B (Fig. 17) be visible, but not accessible from K and N; measure K N and observe angles N K B, K N B and K B N, or two of them, then $KB = KN$ (part of the tangent E P) $\times \dfrac{\sin KNB}{\sin KBN}$

$$= NB \frac{\sin KNB}{\sin NKB} \text{ or } NB = KN \frac{\sin NKB}{\sin KBN}.$$

A base or bases K B and N B have now been determined and in the triangle K P B,

$$KP = KB \frac{\sin KBP}{\sin KPB} \text{ and } PB = KB \frac{\sin BKP}{\sin KPB}$$

and in the same way, in the triangle

$$N P B, N P = N B \frac{\sin P B N}{\sin N P B}.$$

Now, having solved the triangle A P B, it becomes necessary to determine either, if the radius be fixed, the true positions of the tangent points, or if the starting point of the curve be fixed, the necessary radius to make the curve fit in exactly.

If the radius of the curve be fixed,

$$\text{Tangential Distance } A P \text{ or } P B = \text{Radius} \times \text{Cotangent } (\tfrac{1}{2} \text{ apex angle } A P B).$$

Then, as K P or N P is known, the difference between either of them and A P will give the length K A or N A to be set out from K or N in order to fix the starting point A. The finishing point B can be similarly fixed.

If the tangential distance is known exactly.

$$\text{Radius} = \text{Tangential Distance} \times \text{tangent } (\tfrac{1}{2} \text{ apex angle } A P B).$$

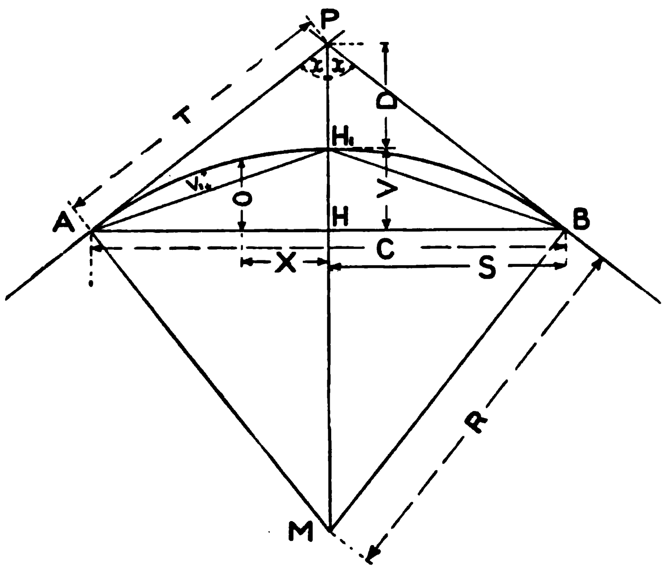


FIG. 18.

The following formulæ (Fig. 18) will also be found useful when setting out a curve by ordinates from a chord, the method of doing which will be explained later.

$$D = \text{Radius} \times \{(\text{cosec } \tfrac{1}{2} \text{ apex angle}) - 1\}.$$

Where D = Distance of middle point on curve from the apex.

$$S = \text{Radius} \times \text{Cosine } (\tfrac{1}{2} \text{ Apex Angle}).$$

where

$$S = \tfrac{1}{2} \text{ chord A B joining tangent points.}$$
$$V = \text{Radius} \times \text{Coversine } (\tfrac{1}{2} \text{ Apex Angle}).$$

where

$$V = \text{Versine of curve.}$$

The number of chords in any curve is given by the expression

$$\frac{5400 - \frac{1}{2} \text{ apex angle in minutes}}{\text{Tangential angle in minutes}}.$$

Tangential Angles.

Let A H (Fig. 19) be any chord.
A P be the tangent at A.
Then angle P A H is the tangential angle for the chord A H.

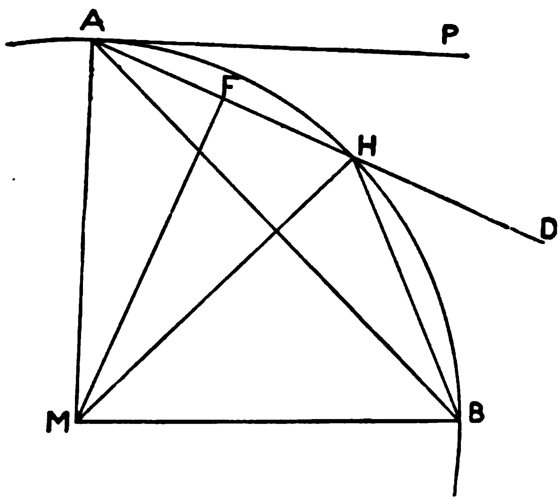


FIG. 19.

Now if H B is another chord of equal length to A H, the angle A M B will be twice the angle A M H, and hence the angle P A B will be twice the angle P A H. For the second chord H B, therefore, the tangential angle will be twice the first one and so on.

Now suppose A H is a known length, say 1 chain, and F is the middle point, then $\frac{\text{A F}}{\text{Radius}} = \sin \text{AMF} = \sin \tfrac{1}{2} \text{ A M H} = \sin \text{P A H}$ (the tangential angle).

Hence $\sin \text{(tangential angle)} = \tfrac{1}{2} \frac{\text{chord}}{\text{radius}}$.

Now the tangential angle in minutes $= 1719 \times \frac{\text{arc}}{\text{radius}}$ but as for the length of chord used we may consider the arc as equal to the chord, we may write this expression.

$$\text{Tangential angle in minutes} = 1719 \times \frac{\text{chord}}{\text{radius}},$$

where chord is not greater than $\frac{\text{radius}}{20}$.

Setting out Curves by the Method of Tangential Angles.

Place the centre of the theodolite vertically over the point A (Fig. 20). This is usually accomplished by hanging a plumbob from the centre of the theodolite and shifting the legs of the instrument until the plumbob hangs vertically over the centre of the tangent peg and in close proximity to it.

More elaborate instruments are provided with a movable locking plate placed between the stand and the underside of the theodolite, which allows of the instrument being traversed about $\frac{3}{4}$ inch in any direction measured from the centre, so that the plumbob may be brought approximately over the centre of the peg by moving the legs, the final adjustments being made with the traversing plate.

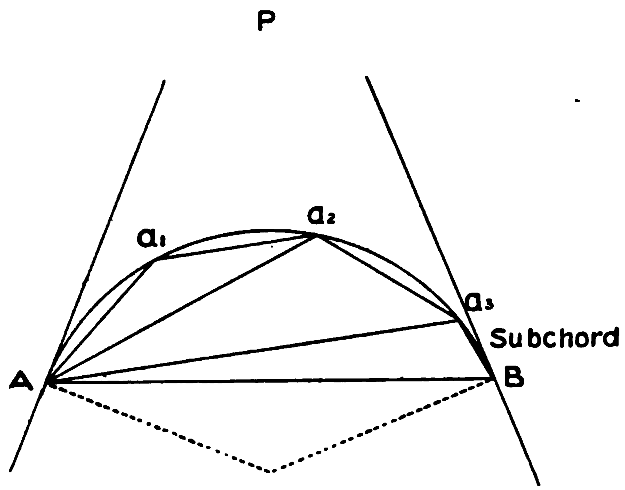


FIG. 20.

As in high winds the plumbob is liable to considerable oscillations and departure from the vertical even when protected by screens, a further elaboration, which the writer has found to be of great value, is sometimes provided by boring out the vertical axis of the theodolite and inserting a vertical telescope of low power in the axis. The telescope is provided with cross-hairs and a diagonal eyepiece, so that the head of the peg and of the central nail driven into it can be observed. The cross-hairs can be brought to coincide with the central mark and must remain coincident with that mark while the theodolite is rotated about its vertical axis, i.e. the theodolite must be level.

In open country great refinement of position is unnecessary, but in populous districts, in tunnelling and elsewhere, even greater accuracy than that of a peg with a nail driven into it may be required, and the use of a metal plate, finely marked, and set in masonry or concrete, may be necessary.

Having brought the vertical centre of the theodolite to coincide with the tangent point, unclamp both the upper and lower circles, bring the zero on the principal vernier to coincide with 360° on the circle, clamp the upper circle and direct the telescope roughly towards the point P; transit the telescope, that is

turn it over in its bearings, sight a pole or other mark on the tangent line, clamp the lower circle and with the tangent screw bring the cross-hairs on the diaphragm to coincide exactly with the pole or other mark. Then re-transit the telescope, when, if the point P be visible, the cross-hairs should bisect it.

If a Y theodolite be used, proceed as before, but, instead of transiting the telescope, lift it out of the Y's, turn it end for end and re-insert it in the Y's to read the back mark, with which the cross-hairs must coincide. When both plates are clamped, lift the telescope from its Y's again, turn it end for end, re-insert in the Y's and proceed.

Where an Everest theodolite is employed, direct the eye-piece end of the telescope towards P, bring the vernier to coincide with 180° on the circle, clamp the upper plate and direct the telescope towards the back pole or mark; clamp the lower circle and, rotating the telescope by means of the tangent screw on the lower circle until the cross-hairs in the telescope coincide with the mark on the tangent line and bisect it, unclamp the upper circle and rotate the verniers through 180° until the vernier which before read 180° now reads 360°, and clamp.

The clamping must in every case preceed the final adjustment, which must be made with the tangent screw.

Having brought the line of collimation of the telescope to coincide with the tangent line, unclamp the upper plate of the theodolite and set the vernier to the first tangential angle, which, for a 20 chain curve, is 1° 25′ 57″. Lay out 1 chain, and at the end of the chain and at the point intersected by the cross-hairs in the theodolite, put in a peg and drive a nail into the head of the peg at the required length and point of intersection.

Unclamp the upper plate and lay out the angle 2° 51′ 54″, or twice the tangential angle, lay out one chain from the last peg, and where the chain end meets the point of intersection of the cross-hairs insert a peg and proceed as before, the next angle to be laid off being 4° 17′ 51″.

Now suppose the curve is 26·529 chains in length, there are 26 full chain chord angles of 1° 25′ 57″ or 37° 14′ 42″ in all. The last chord has a length of 52·9 links, the angle for which will be, $85{\cdot}95' \times \frac{52{\cdot}9}{100} = 45'\ 28''$, making a total angle from the start of 38°, and if the last point be laid out with this chord and angle, it should check on to the finishing point already fixed.

With the theodolite in common use it is not possible to read more nearly than 30″ of arc.

It frequently happens that all the chord ends on the curve to be set out cannot be seen from the tangent point. Say only 6 chords are visible. Then having fixed the sixth peg and nail therein carefully, unclamp both plates, move the theodolite and set it up accurately over the new station at the end of the sixth chain; with the telescope looking forward, fix the vernier to read 360°, transit the telescope and intersect the mark at the tangent point. Clamp the bottom plate, re-transit the telescope, unclamp the top plate, and bringing the vernier to read 10° 1′ 39″, the tangential angle for seven chords, proceed as before.

An alternative method is to move the theodolite to the sixth peg and with the telescope pointing forward, bring the vernier to read 351° 24′ 28,″ which is 360° less 8° 35′ 42″, the tangential angle for 6 chords. Clamp the upper plate, transit the telescope, and intersect the tangent mark; clamp the bottom plate, re-transit the telescope, and bring the vernier to 360°. The line of collimation of the telescope will now be tangential to the curve at the sixth peg, and angles may be laid out as before, beginning with 1° 25′ 57″.

If a second obstacle occurs, proceed as before, using the mark on the sixth peg as the tangent point.

Should it be desirable to set out the curve from the tangent point B, instead of from A, the first tangent angle for a 20-chain curve will read 358° 34′ 3″, the second 356° 8′ 6″, and so on.

In changes of position of the theodolite the tangential angle for the number of chords already laid out must be added to, instead of being subtracted from, 360°.

It is advisable, before commencing to lay out a curve by this method, to draw up a table showing the tangential angle corresponding to each chord set down.

The preceding formulæ are equally applicable where a 20 metre or a 100 feet chain is used.

Degree Curves. In American practice the unit of measure is not the radius but the degree, where one degree measured at the centre of the circle is subtended by an arc of 100 feet, and the tangential angle in a degree curve = $\frac{1}{2}°$.

The unit of measurement in this case is the degree not the radius; in other respects the formulæ already given apply.

Example.—Let the apex angle A P B (Fig. 16) equal 104°, then half apex angle = 52°.

Let the curve be one of 4°, the tangential angle being 2°, or 120 minutes.

Then

$$\text{Radius} = \frac{1719 \times 100}{120} = 1432 \cdot 5 \text{ feet.}$$

More exactly, if the curve is one of $n°$, radius $= 50 \times \text{cosec}\frac{n°}{2}$.

This formula should be used when n is greater than 4°

$$\begin{aligned}\text{Tangential Distance} &= 1432 \cdot 5 \cot 52° \\ &= 1432 \cdot 5 \times 0 \cdot 7813 \\ &= 1119 \cdot 2 \text{ feet.}\end{aligned}$$

Method of Tangential Angles with two Instruments without Chain. Let the two theodolites be set up, one at A and the other at B (Fig. 21). Clamp the upper plates of both instruments to read zero or 360°.

Bring the line of collimation of each to coincide with the tangent line and clamp the lower plates, then, for a 20-chain curve with a length of curve of 26·529 chains, set off at A 1° 25′ 57″ and at B 360° − 36° 34′ 3″ the reading being 323° 25′ 57″ The angle 36° 34′ 3″ is made

up of $1° 25' 57'' \times \frac{52 \cdot 9}{100} = (45' 28'')$ for the last 52·9 links before the last chain end of the curve is reached and of $1° 25' 57'' \times 25$ ($= 35° 48' 45''$) for the 25 full chain chords back to the first point to be laid out. The point at which these lines intersect is on the curve. At two chains from A the respective readings will be $1° 25' 57'' \times 2$ ($= 2° 51' 54''$) and $323° 25' 57'' + 1° 25' 57''$ (or $324° 51' 54''$) respectively and so on.

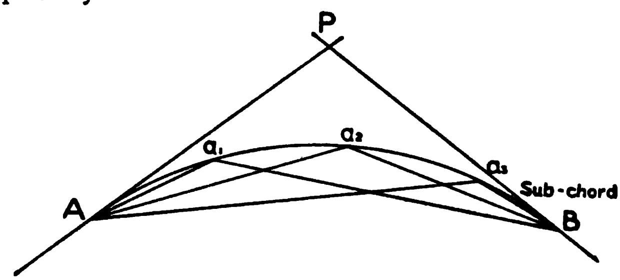


FIG. 21.

It may be noted that any two angles which together make up one of the angles at the base of the triangle A P B will intersect a point on the curve.

In open country, or where the theodolite must be shifted frequently, the use of one instrument and a chain is to be preferred, but, where a deep gully must be crossed, the use of two theodolites will generally be preferable.

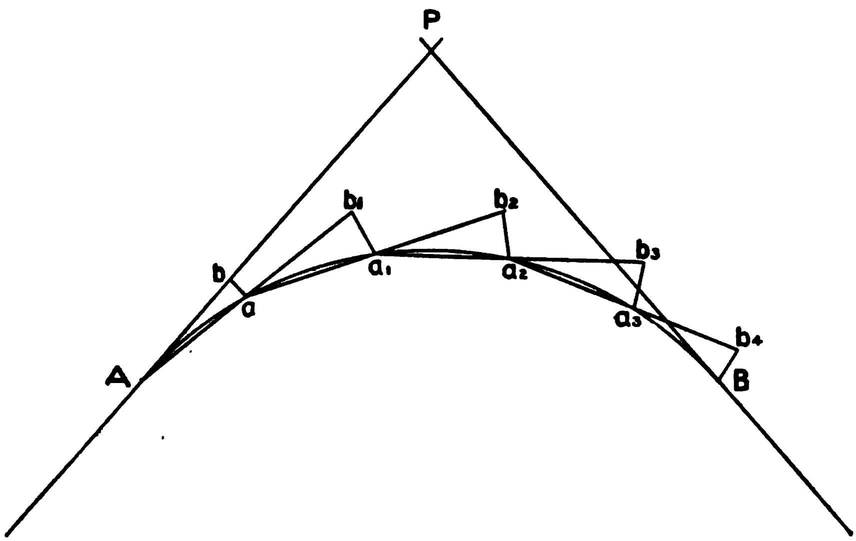


FIG. 22.

Method of Deflection Angles.

Laying out curves with theodolite, chain and deflection angles.

As in the previous methods, set up the theodolite at A (Fig. 22) and bringing the line of collimation of the telescope

to coincide with the tangent line A P, lay off the tangential angle PAa, which is equal to, for a 20-chain curve, 1° 25′ 57″. Chain the distance Aa, and insert a peg at the point of intersection, then move the theodolite to *a*, bring the vernier to zero, transit the telescope and observe point A, lay off the angle 2° 51′ 54′ which is twice the tangential angle, retransit the telescope, chain from *a* to a_1 and, where the chain end is intersected by the line of collimation of the telescope, put in a peg. Next move the theodolite to a_1 and repeat, the angle used being double the tangential angle.

This method is slow and less accurate than the methods previously described but there may be occasions when its use is advantageous.

From the formula tangential angle $= 1719 \frac{\text{chord}}{\text{radius}}$ it is obvious that the tangential angle varies directly as the chord and inversely as the radius. Thus if a chord is a constant of 1 chain the angle for a 10-chain curve is 171·9 minutes, for a 20-chain curve 85·95 minutes and for an 80-chain curve 21·4875 minutes.

The radius being constant the angle for a ½-chain chord is half that for a 1-chain chord and proportionately for any other multiple or fraction. Also if the radius and chord be varied proportionately the angle remains the same, from which it will be seen that it does not matter whether a 66 feet, a 100 feet, or a 20 metre chain be used so long as the radius and the chord are both measured in the same units.

Where an Instrument other than a Theodolite is used.

Curves may be laid out, though with less accuracy, with a sextant, or, with still less accuracy, with a prismatic compass.

The procedure is identical with that already described.

When the curves are short and the radius exceeds 20 chains, or, for filling in short lengths when pegs have been lost, either of these instruments (sextant or prismatic compass) may be used with some advantage, but they should not be employed for sharp or long curves when exactitude is required.

The ordinary 5″ theodolite with verniers is not capable of reading an angle with a greater degree of accuracy than to the nearest 30 seconds which is represented by 0·192 foot in 20 chains. If the sextant be used the probable error is at least four times as great and with the prismatic compass still greater. These probable errors are not, however, at all prohibitive if the distance to be measured is short.

Laying out Curves with Chain and Poles.

By Offsets.—From A (Fig. 22) lay out 1 chain, or for small curves ½ or ¼ chain along the tangent line in the direction of P, put in an arrow at *b* at the end of the chain.

The offset from this point to a point on the curve 1 chain distant from A will be $\frac{\text{chord}^2}{2\ \text{radius}}$ = for a 20-chain curve $\frac{66^2}{2640} = 1 \cdot 65$ ft.

Provide a convenient piece of wood, a lath will do, 3′ 6″ in length, cut a notch near one end which will fit on to the arrow placed at *b*, another notch

at 1·65 feet from the first notch and a third at double this distance or 3·3 feet from the first.

Fit the notch to the arrow at *b*, move the chain until it is opposite the second notch at *a* and where the end of the chain coincides with the second notch insert an arrow in the notch.

This arrow will be the first point on the curve and must be replaced by a peg.

Continue the line A*a* to b_1, 1 chain from *a* and insert an arrow.

The offset from b_1 will be $\frac{\text{chord}^2}{\text{radius}} = \frac{4356}{1320} = 3 \cdot 3$ feet.

Apply the end notch to the arrow at b_1, draw the chain to the third notch on the piece of wood and into this notch insert an arrow, to be afterwards replaced by a peg.

Next extend the line aa_1 to b_2, insert an arrow and proceed as before until the last full chain end is reached.

Supposing the length of the final chord to be 52·9 links, the offset will be $\frac{3 \cdot 3 \times 52 \cdot 9}{100} = 1 \cdot 74$ feet and the point reached should coincide with the tangent point B.

If there is a material divergence from accuracy at B, say more than 3 inches, range the curve backwards from B, starting with a chord of (in the illustration) 52·9 links, until a point is reached where the two sets of pegs are in close proximity, or to the apex of the curve arc. If no local error be discovered, remove the first set of pegs up to the point reached from B.

Ordinates measured from chord A B or from two chords A H_1, H_1 B, or from four, eight, etc. chords (Fig. 18).

Method by Ordinates from Main or Subsidiary Chord.

V = versed sine of curve = R (coversin x)

or

$V = R - \sqrt{R^2 - (\frac{1}{2} C)^2}$

O = any ordinate measured from the chord A B.

$O = \sqrt{R^2 - X^2} - (R - V)$ where X is the distance of the ordinate O from the centre of the chord A B.

Example.—Where the radius of the curve is 20-chains or 1320 feet. Suppose the length of the chord to be 24·626 chains.

The angle A M B at the centre is 76°.

Half the angle of intersection of the tangents = x = 52°; then:—

$$V = R\ (\text{Coversin } x) = 1320 \times 0 \cdot 21199$$
$$= 279 \cdot 83 \text{ feet}$$
$$= 4 \cdot 24 \text{ chains.}$$

In this case it is obvious that the versed sine and the other ordinates will be too long to be measured conveniently. To get rid of this difficulty one of two courses may be pursued, namely (*a*) (Fig. 18). Range in the chord A B and place a peg at the centre of its length. Range the line H H_1 representing the

versed sine at right angles to A B $4 \cdot 24$ chains to H_1 at the apex of the curve, put in a peg and join A H_1, H_1 B pegging the lines out in chain lengths and inserting a peg at the centre of each. A H_1, H_1 B each equal $\sqrt{\frac{\text{chord}^2}{2} + \text{perp.}^2} =$ $\sqrt{(12 \cdot 323)^2 + (4 \cdot 24)^2} = \sqrt{151 \cdot 62 + 17 \cdot 98} = 13 \cdot 023$ chains $= 859 \cdot 5$ feet

Then with R = 20 chains and chord = $13 \cdot 02$ chains

$$V_1 = R - \sqrt{R^2 - (\tfrac{1}{2}\, A H_1)^2} = 20 - \sqrt{400 - 42 \cdot 38} = 1 \cdot 09 \text{ chains.}$$

a dimension which is easily dealt with. The subsequent procedure to find the ordinates is as already described.

The second method (*b*) is:—Set out two or more lines parallel to A B and a convenient distances from it (Fig. 23) in the example $1\frac{1}{2}$ and 3 chains distant.

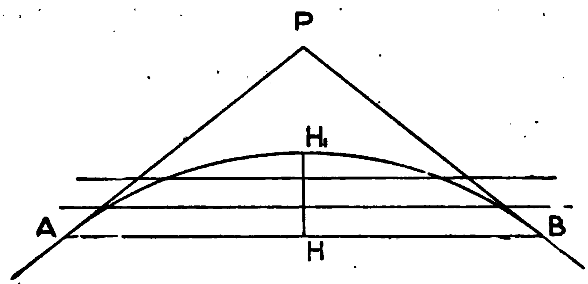


FIG. 23.

Lay out the perpendicular H H_1 and from the points of intersection of the two subsidiary chords, with the line H H_1 chain in each direction, inserting pegs at each chain end, then the distance from the chord which is nearest the circle to H_1 will be $4 \cdot 24 - 3 = 1 \cdot 24$ chains. The method to be followed is to calculate the lengths of the ordinates at every chain from the chord A B, but to set them out from the chord which is measured nearest the curve at that point.

Thus the ordinate from the chord A B at 1 chain from the centre will be (Fig. 18)

$$\begin{aligned} \sqrt{R^2 - X^2} - (R - V) &= \sqrt{20^2 - 1^2} - (20 - 4 \cdot 24) \\ &= 17 \cdot 975 - 15 \cdot 76 \\ &= 4 \cdot 215 \text{ chains.} \end{aligned}$$

This would be set off as $1 \cdot 215$ chains from the chord 3 chains from A B.

At 7 chains from the centre the length of the ordinate is $\sqrt{20^2 - 7^2} - (20 - 4 \cdot 24) = 18 \cdot 735 - 15 \cdot 76 = 2 \cdot 975$ chains. This would be set off as $3 - 2 \cdot 975 = \cdot 025$ chain back from the chord at 3 chains from A B.

Care must be taken that the ordinates are truly perpendicular to the chord A B.

By Off-sets inside the Curve.

Another method which may be adopted where the ground inside the curve only is favourable for plotting is as follows:—

Calculate the versed sine of the angle which a chord subtends at the centre of the curve of given degree, or to a given

radius. Then at A (Fig. 24), the point from which the curve springs, set off A b equal to half of this versed sine, and in the direction of the centre of the curve, then measure A E backwards equal to half the selected length of the chord, and produce E to a_1, making $b\,a_1 = b\,$E. Then a_1 is the first point in the required curve. Now, from a_1 set off the full versed sine towards the centre and range A b_1 a_2, making $b_1 a_2 =$ A b_1. This gives a_2 the second point in the curve, and so on till the point of entering the straight is reached.

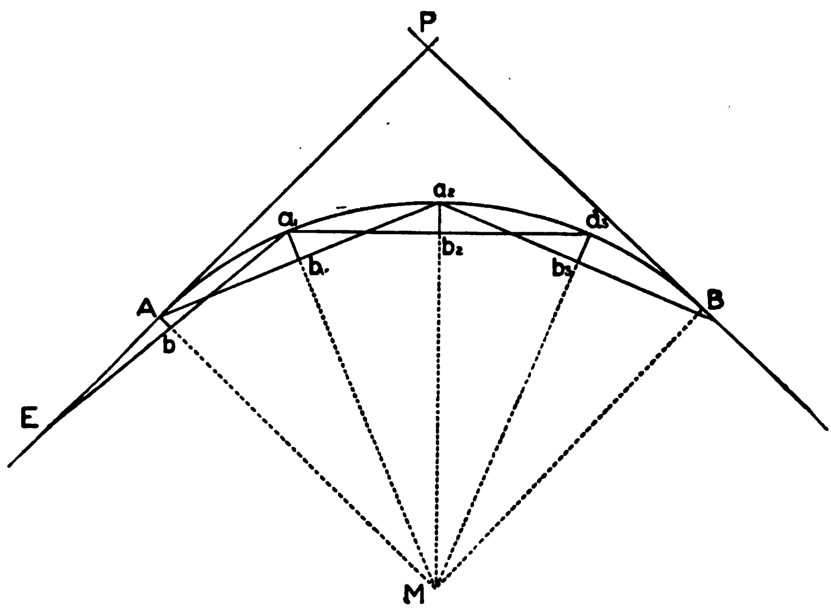


FIG. 24.

Compound Curves. The object of a compound curve is to avoid certain points the crossing of which will involve expense, and which cannot be avoided by a simple curve.

It being assumed that a large scale survey is available on which the curves can be laid down, it will be obvious through which points it is desirable that the line shall run; if this cannot be approximately accomplished with a simple curve, it will be necessary to have recourse to a compound curve.

No part of a compound curve must be of less radius than some fixed minimum depending on the speed at which the curve is to be negotiated.

Taking an example, it is assumed that the inclination of the tangent lines to each other is known, with such accuracy, at any rate, as can be secured by compass bearings taken with the theodolite. Let $x = \dfrac{\text{apex angle}}{2} = 22°$, and let the length of each of the tangent lines be 70 chains, then the radius of a simple curve joining the tangents will be 28·28 chains. If this curve cannot be used, the problem is to join the tangent points by two or more curves passing through such points as $\gamma\,\delta\,\epsilon$. No curve may be of less than, say, 15 chains radius.

There is no positive method of solving the problem, and in all probability it will not be truly soluble without using an unreasonable number of curves, so that it will be necessary to proceed graphically and to fit the curves so as to satisfy the conditions as nearly as possible.

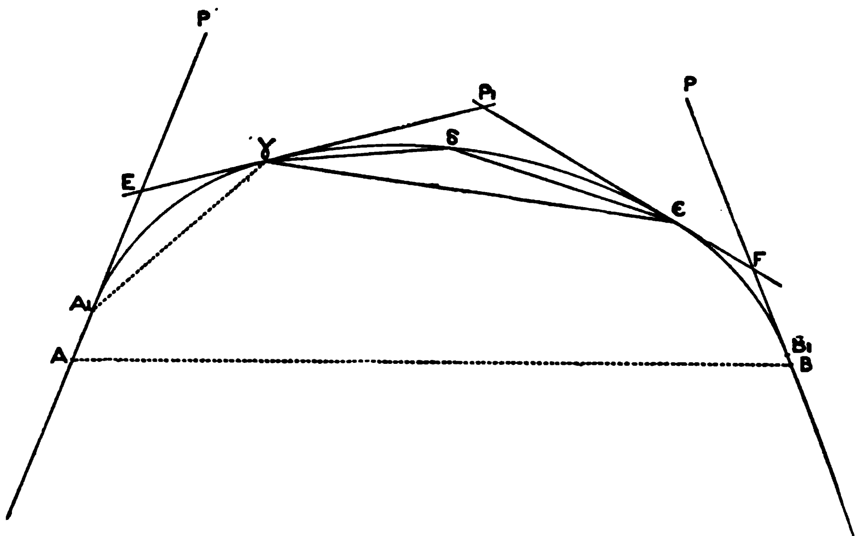


FIG. 25.

The construction may be proceeded with as follows: Join $\gamma\,\delta$ and $\delta\,\epsilon$ and chain each line, also read the angles $\delta\,\gamma\,\epsilon$ and $\delta\,\epsilon\,\gamma$. At γ set out the angle $\delta\,\gamma\,P_1$, equal to $\delta\,\epsilon\,\gamma$, and at ϵ set out $\delta\,\epsilon\,P_1$, equal to $\delta\,\gamma\,\epsilon$. Then the lines $\gamma\,P_1$ and $\epsilon\,P_1$ will be tangential to the curve passing through the points γ, δ, and ϵ.

The radius of this circle may be calculated from the equation: $R = \frac{1}{2}\,\delta\,\epsilon$ cosec $\delta\,\gamma\,\epsilon = \frac{1}{2}\,\delta\,\gamma$ cosec $\delta\,\epsilon\,\gamma$.

As the radius and the directions of the tangents are now known the curve can be set out.

Produce $P_1\gamma$ and $P_1\epsilon$ to cut the main tangent lines A P and B P in E and F respectively; from E measure E A_1, equal to $\epsilon\,\gamma$, towards A and from F measure F B_1, equal to $P_1\,\epsilon$, towards B. Then if A_1 and B_1 coincide with A and B, the problem has been solved, for a curve can be drawn through A and γ tangential to A_1E and γ E, and similarly a curve can be drawn through ϵ and B tangential to ϵ F and F B.

But as A_1 and B_1 will probably not coincide with A and B, it becomes necessary to decide whether the curves obtained are satisfactory, or whether it is better to so alter the curve $\gamma\,\delta\,\epsilon$ that the points A_1 and B_1 are brought nearer to A and B, while the curve $\gamma\,\delta\,\epsilon$ does not diverge too widely from the points γ, δ, and ϵ.

In this respect every separate case must be settled on its merits.

Another method is to start by drawing curves tangential to A P and B P at

A and B and passing through γ and ϵ respectively, and then arrange the best possible curve or curves to join them, thus:—

In the triangle A P B (Fig. 26), join A γ, B ϵ.

From γ lay out angle E γ A equal to P A γ.

Join γ E, making γ E = A E.

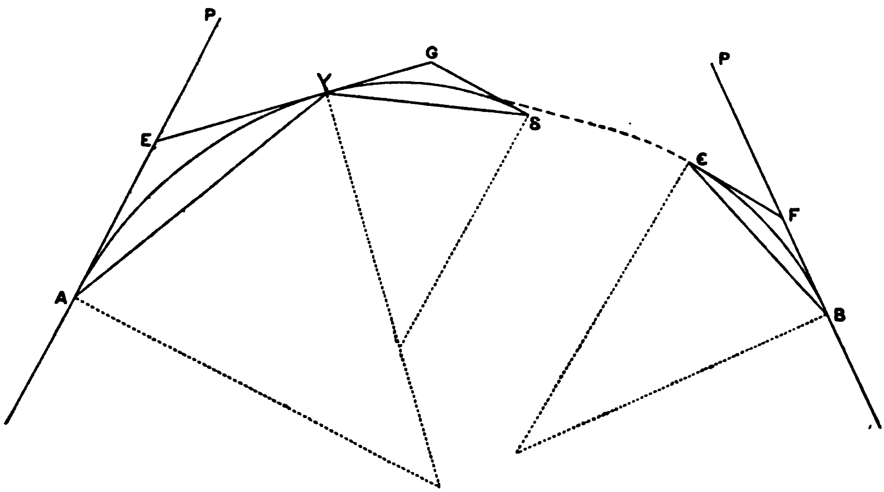


FIG. 26.

Then A and γ are points on the curve, A E is tangential to the curve at A and E γ at γ.

Observe the angle A E γ = say 134°, then $x_{,} = 67°$.

The tangent lines A E, E γ are 11·5 chains in length.

Then the radius of the curve from A to γ will be R = T (tan $x_{,}$) = T (tan 67°); R = 11·5 × 2·35585 = 27·1 chains radius.

Similarly:—

Lay out angle B ϵ F to equal F B ϵ.

Join ϵ F, equal to F B = 7 chains.

Read the angle B F ϵ = 145°, therefore $x_{,,} = 72° \ 30'$.

Then R = T (tan $x_{,,}$) = 7 × 3·17159 = 22·2 chains.

Again:—

Join γ δ.

Extend the tangent line E γ to G, and make G δ = γ G = 7·2 chains.

Read the angle γ G δ = 135° and $x_{,,,} = 67° \ 30'$.

R = T (tan $x_{,,,}$) = 7·2 × 35585 = 16·96 chains.

At this point in the investigation it becomes obvious that the tangent line G δ will fall inside the tangent line F ϵ, that no curve tangential to the two can be inserted, nor will a straight be tangential to both. There is, however, no difficulty in adjusting suitable curves which will commence at A, pass in close proximity to γ, δ, and ϵ, and terminate at B. The exact location of each curve must depend upon the proportional value put upon exact adherence to the three

points γ, δ, and ϵ. In Fig. 26 a simple solution, where the least value is given to δ, is shown in dotted lines.

As the surveyor will have laid down on plan the points to be traversed, when the curves at the ends have been laid out it will be apparent whether the third point can be cut exactly or whether some small adjustment is necessary.

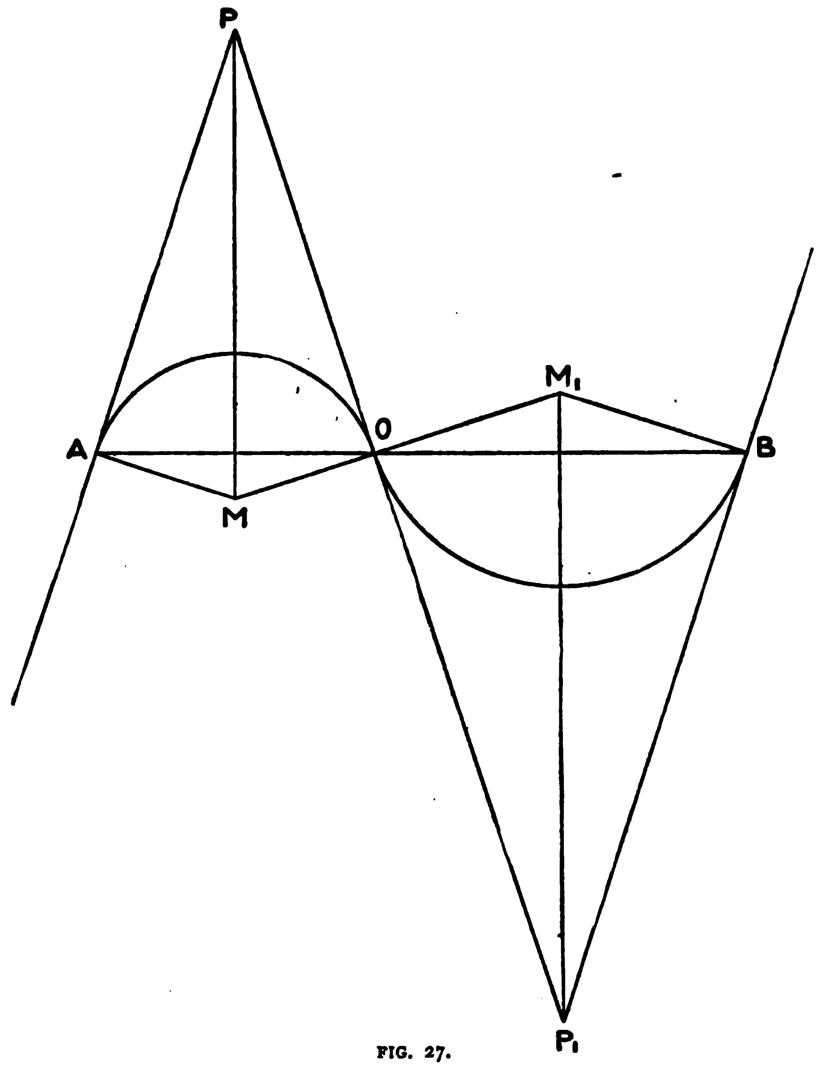


FIG. 27.

Reverse or Serpentine Curves.

In Fig. 27 let A P, B P_1 be the principal tangents; join A B and divide it into two parts, proportional to the curves to be used, at O.

Bisect A O and O B and raise perpendiculars from each to

cut the tangent lines A P at P and B P_1 at P_1, then the angles A P O, B P_1 O will be the angles at the intersections of the tangents.

Measure the chords A O and O B and the angles P O A, P_1 O B, or the angles A O M and B O M, whichever may be more convenient. Then the angles at the apexes and centres can be calculated, also the lengths of the radii and tangents.

This construction is only applicable when the tangents A P and B P_1 are parallel, as otherwise the point of reverse curvature will not lie on the line A B. Example: where the curves come into tangent at O.

Let the chord A O = 30 chains, and B O = 40 chains, and the angles P A O, P O A, P_1 O B, P_1 B O = 72°, then the apex angles A P O and B P_1 O = 36°, and the centre angles A M O, B M_1 O = 144°.

The radii are 15·76 and 21·02 chains, and the tangents 48·54 and 64·72 chains in length, respectively.

To introduce 3 chains of straight between the two curves.

Along the tangent line A P measure 1½ chain from A towards P, and from B 1½ chain towards P_1 along B P_1, and from O measure 1½ chain towards both P and P_1.

Then the angles remain the same, but the lengths of the tangents are 47·04 and 63·22 chains respectively, and the radii are

$$R = T \tan x = 47{\cdot}04 \tan 18^\circ = 15{\cdot}28 \text{ chains}$$

and

$$R_1 = 63{\cdot}72 \tan 18^\circ = 20{\cdot}70 \text{ chains.}$$

Both the radii and the tangents are proportional to the chords.

Laying Rails after the Embankments and Cuttings are finished. The work of laying the rails on the finished road surface, about 18 inches above the top of embankments and bottom of cuttings, is performed by means of setting out abscissæ and ordinates along the line of chords. These are taken out from tables of ordinates calculated for abscissæ at 5 feet intervals. Vide "The Field Practice of laying out Circular Curves for Railroads," by J. C. Trautwine, C.E., or "Railway Engineering," by C. B. Smith, M.E.

Curves of Adjustment. On the straight, both lines of a railroad are laid at the same transverse level, but once a curve is entered on it is found necessary to give the "outer rail" a super-elevation over the "inner rail," proportional to the radius or degree of the curve, and the maximum rate at which ordinary traffic passes over it.

This super-elevation is given in order that the resultant of weight and centrifugal force shall remain at right angles to the road. Although with high speeds and sharp curves theory may indicate a larger amount as necessary, in practice 6 inches is adopted as a maximum.

The best way is to gradually increase both elevation and curvature, as this super-elevation must be "gained" and "reduced" gradually to avoid jerk or strain when entering or leaving. It is obvious that some adjustment is necessary to meet the case, and this is arrived at by adopting "curves of adjustment." One of the following methods is now adopted in all European or American railway systems.

(1) A succession of short pieces of curve of ever-decreasing radius till the maximum super-elevation required is reached.

(2) A form of spiral curve, being a modification of (1).

(3) A modified (*a*) quadratic, or (*b*) cubic parabola.

Method (1) is tedious, clumsy, and inexact, when, as is sometime done, "the trackman" is allowed to adjust the first 100 feet or so. Sometimes lengths of 30 feet are laid out to successive degrees of curvature 1°, 2°, 3°, etc., when the theodolite must be shifted for each 30 feet.

For method (2) vide "Railway Spirals," by Searles.

Method (3*a*) is the "Holbrook Spiral," in which correct horizontal alignment is sacrificed for a supposed refinement in the vertical adjustment.

Method (3*b*) is described, with equations of the "cubic parabola," in the January and February numbers (1890) of the "Engineering News" and in the supplement to "Krohnke's Handbook."

The curve required for a suitable transition curve is one which, starting from the tangent point on the straight, with an infinite radius, or degree of curvature zero, has a degree of curve at succeeding points, in direct proportion to the distance from the tangent point, till it joins and becomes tangent to the main curve.

Now the cubic parabola, where $y = (f)\ x^3$, approximates very nearly to these conditions.

As tangents and the main curve are fixed in position by construction, adjustments made by the plate-layer, and by leading off for the first hundred feet or so with a curve of larger radius than the main curve, merely amounted to flattening the ends at the expense of the central portion, which has to be made sharper, forming more or less of elbows in the track.

When a suitable easement or adjustment curve is used, this can be avoided by sharpening the whole main curve slightly, or by changing the position of the tangents, transferring the position of the main curve inwards to an amount fixed by calculation, or by moving the tangents outwards to a similar amount—either of which alterations will admit of the introduction of a suitable easement curve. It is beyond the scope of this treatise to fully investigate the easement curves, above referred to, or to give the tables required for working them out, and the reader is referred for further information on this subject to the following works: "Railway Engineering," by Mr. C. B. Smith, M.E.; "Field Practice of Laying Out Railroad Curves," by Mr. John C. Trautwine, C.E.; and "Handbook for Laying Out Curves," by Krohnke. The latter is the latest work.

The following easy method, due to the late Mr. G. J. Morrison, of Shanghai, is worthy of note.

In the first place, let all elastic curves, or entrances to the main curve, be of the same length, say either three chains of 66 feet each, or two chains of 100 feet each. The springs of an engine will allow of one rail being elevated 6 inches in a length of 132 feet, and more easily in 198 feet, or three chains, without straining the machinery.

When a circular curve leaves a straight line, the off-set from the straight to the curve at the end of a chain is a definite amount depending on the radius, say, for a 5 chain curve it is 6·6 feet.

Before setting out any curve, let a supplementary straight be laid out parallel to the true straight, and at a distance from it inwards from the tangent equal to one third of the off-set of the circular curve, in one standard length. Thus at a distance of 99 feet, adopting 66 feet as a standard, the off-set $\left(\frac{\text{chord}^2}{2\ \text{radius}}\right)$ for a 5 chain curve is 14·85 feet, and the distance of the supplementary straight from the true straight 4·93 feet. The distance for a 10 chain curve is 2·46 feet, for a 20 chain curve 1·23 feet, &c.

Let A B, B C (Fig. 28) be the two straights of the railway. Set out the lines D E, E F, parallel to them, and (for a 5 chain curve) at a distance of

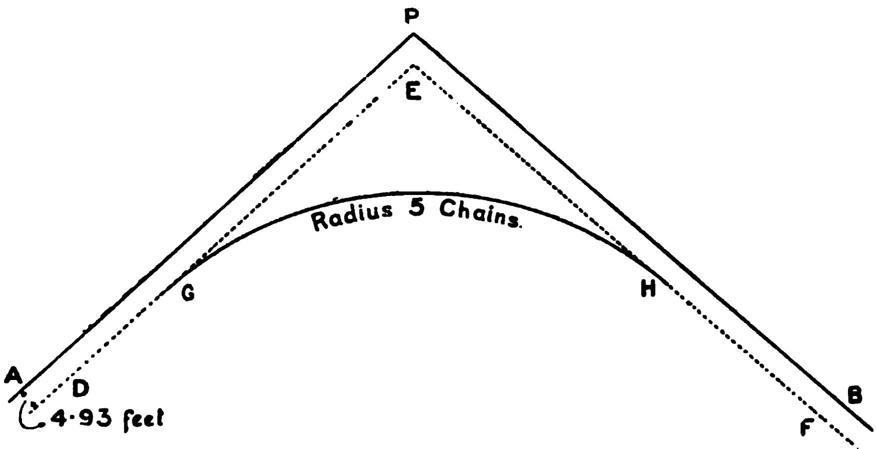


FIG. 28.

4·93 feet from the original straight. The lines D E, E F are to be taken as the tangents, and not the lines A P, B P, and the circular curve G H is to be laid out, joining them by the most convenient method.

By this means we have a straight line A K (Fig. 29) and a circular curve G P both laid out as easily as if no elastic curve had to be considered.

Put in pegs at K and G, and also one at N halfway between them. Put in a peg at L, 99 feet back from K, and one at M, 99 feet on from G. The curve passing through L, N, M is a cubic curve tangent to the straight at L, and tangent to the circle at M. If the super-elevation be gradually increased from L to M, there will be the proper elevation at each point in the line, and the train will enter the curve without the slightest shock.

If intermediate points have to be set out in the elastic curve, as will be the case in a tunnel, or near bridges or other masonry, all that has to be done is to set off off-sets from the straight of the railway on the basis of the off-set K N, and varying as the cube of the distance from L.

Only one other point has to be noted, namely, that this setting out must be adopted at the original staking of the curve. It will not do to treat it as a refinement to be introduced at the end, when stakes are being put in for rail laying.

A practical method of introducing easement curves so as to improve existing lines is described in a paper by W. H. Shortt, Proc. Inst.C.E., Vol. clxxvi. p. 97.

Replacing Lost Stakes.

To replace stakes in a curve if lost, or to add intermediate pegs :—

Let O = the ordinate from centre of chord to curve in feet.
N = length of chord in feet.
R = radius of curve in feet;

then $O = R - \sqrt{R^2 - (\frac{1}{2}N)^2}$.

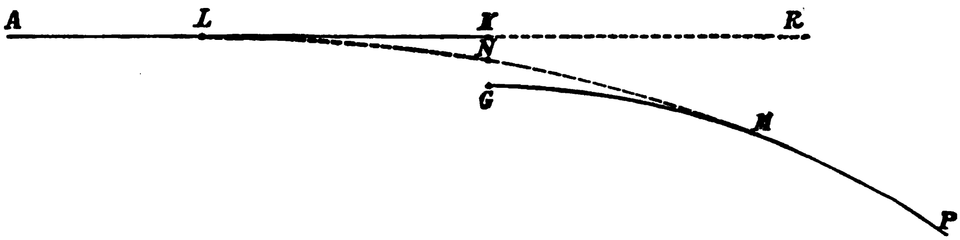


FIG. 29.

Or by another method, the off-set from a tangent $= \frac{\text{chord}^2}{2\,\text{radius}}$, or from a chord produced $= \frac{\text{chord}^2}{\text{radius}}$;

from these formulæ it is plain that

(1) Off-sets vary inversely as the radius.

(2) Off-sets vary directly as the square of length of chord. (*Vide* Points and Crossings, Appendix G).

Example of Survey Work when Laying Out a Tunnel.

As an example of the survey work necessary in connection with the construction of a tunnel, the following description of the setting out work which was undertaken at the construction of the Fairlie Tunnel is inserted.

This tunnel now forms a part of the Glasgow and South Western Railway system, and connects a point on the line two miles north of West Kilbride with the north end of the village of Fairlie, where the railway runs on to the pier.

The tunnel is 971 yards long, the greater portion being on a curve, with a radius of 1 mile 6½ furlongs and a mean gradient of 1 in 100. Two shafts were sunk on the line of the tunnel so as to facilitate progress and to enable the excavation to be carried on from four points simultaneously. These shafts were so placed as to divide the lengths into nearly equal sections, and on the completion of the work they were used as ventilators.

The setting out of a tunnel from open ends only is a matter of little or no difficulty owing to the fact that long base lines from which to work are generally

available. When, however, the setting out has to be carried on from shafts as well, the greatest care must be exercised in transferring the lines tangent to the curve from the surface to the level of excavation. These lines are necessarily limited in length by the dimensions of the shaft so that a very slight deviation of one of the points to one side or the other would cause a very considerable error to accumulate in half the length of a section.

Fig. 33 shows the arrangement of lines which was adopted for setting out the work in this particular case.

In setting out the centre line on the surface it will be seen from the figure that two intermediate tangent lines B C and C D were run, in such a manner that their points of contact with the curve were approximately at the centres of the shafts.

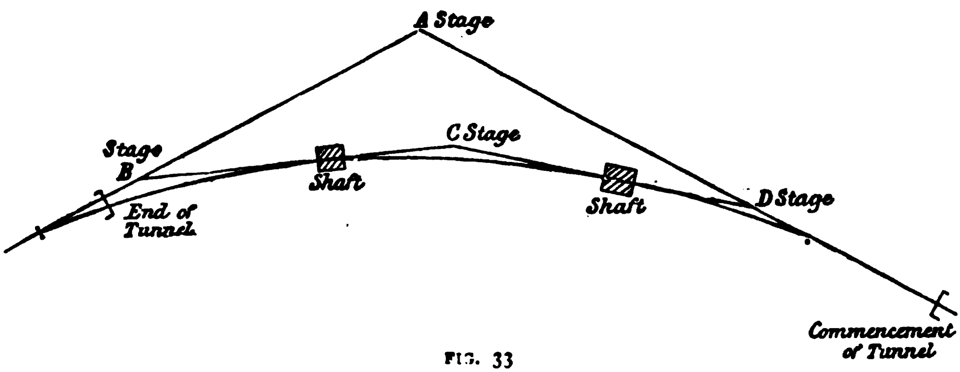


FIG. 33

In order to adjust these points finally before setting out the tunnel, stages of various heights (according to the nature of the ground) which were visible from one another, were erected over the points of intersection B, C and D. All necessary adjustments to ensure that the lines were tangential to the curve at or very near to the centre of length of each shaft were then made to both the angles and distances. The intermediate tangent lines were run across the shafts, and the points at which the lines cut the edges of the shafts were marked by small notches cut in metal plates overhanging the edges of the shafts (*vide* Fig. 34).

In order to get these notches very accurately fixed, the theodolite was set up close to the shaft and adjusted exactly in the line of the tangent, by sighting on one stage, transiting, and observing whether the line from the one stage produced exactly cut the other stage, moving the instrument laterally, and repeating until the line from either stage to the theodolite did, when produced, exactly intersect the other stage, and the three points were therefore truly in the same straight line. After this was accomplished, nothing remained but to fix the position of the notches, which could be done very minutely with the instrument close at hand. By this means a line tangent to the curve at the centre of the shaft was fixed.

The transferring of the tangent line to the bottom of the shaft was accom-

plished by hanging heavy cast iron weights from the notches overhanging the edges of the shaft by means of copper wires as shown in Fig. 34, the weights being immersed in buckets of water. These weights were of about 40 lbs. each and were constructed with feathers projecting from the body of the weight to prevent any rotation in the water. Fig. 35 shows the form of these weights. The feathers, together with the lateral resistance of the water, kept the weights in a steady position.

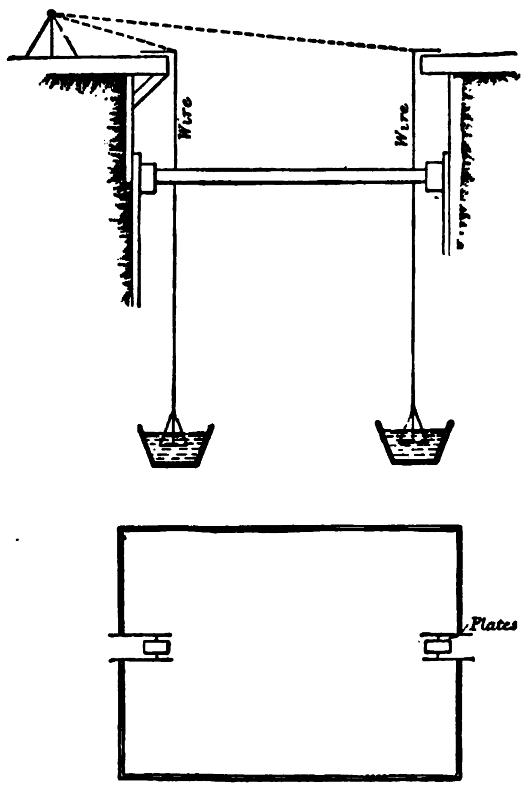


FIG. 34.

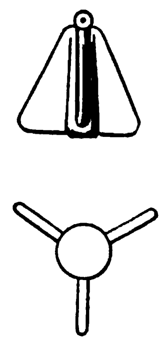


FIG. 35.

In this way the direction of the tangent line was accurately defined at the bottom of the shaft, and the exact tangent point could be found by a measurement between the wires.

Points on the curve in the tunnel could now be easily determined by prolonging this tangent line, and taking offsets calculated according to the distances of the several points from the tangent point at the centre of the shaft. It was, of course, immaterial whether the tangent point was exactly at the centre of the shaft or not, so long as its position, corresponding to that above ground was accurately known.

The method used for prolonging the tangent lines in the shafts was to fix

points ahead, in line with the wires, by eye; for this purpose a small lamp was used, the flame being lined in very carefully with the wires and an offset measured from it, such offset being calculated according to the distance of the lamp from the tangent point.

The question naturally arises as to the procedure when the points which had to be determined on the centre line were at such a distance from the tangent line that the latter could not be continued without coming into contact with the sides of the tunnel. In the main workings this difficulty did not arise, since, owing to the flatness of the curve, the maximum offset was in no case greater than half the width of the tunnel, consequently the tangent line did not meet the sides. It had, however, to be dealt with in the headings which were about 6 ft. by 6 ft. In such cases, when the tangent line became exhausted, another tangent was run as shown in Fig. 36, this process being repeated when necessary.

The *modus operandi* was as follows:—

When the tangent line approached the side of the tunnel as at B, equal offsets were measured from A and B, and the new tangent line D C was run

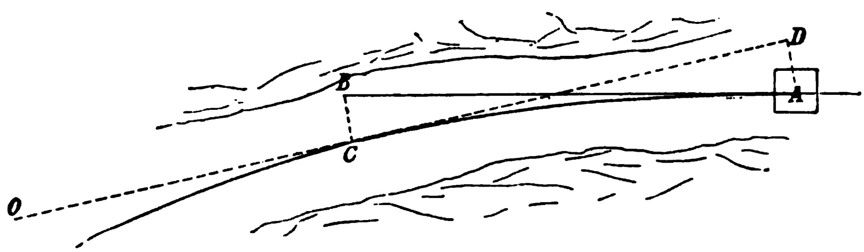


FIG. 36.

forward in the direction of O. It must be pointed out that the method of setting out the offsets B C and A D at right angles to A B is not theoretically correct. To obtain a truly accurate result A B and C D must both be equal to A C. In the case under consideration, however, as in all practical cases of curves in tunnels, the length of chord which can be used is so small compared to the radius (in this case R = 9570 ft. and the longest chord which can be got in is only 239 ft.) that the error introduced by setting out B C and A D at right angles to A B is too small to be measured.

Points were fixed by nails in wooden plugs driven into the roof of the heading, and when any lines required to be extended, strings with weights at their ends were hung from the nails, and these being illuminated by lamps with suitable shades could be sighted and other points fixed ahead.

Greater accuracy can be obtained by setting out the curve with a theodolite, using the method of tangential angles as previously described, and by this means nearly twice the length of curve can be set out from one tangent line as the instrument does not require to be moved until the line of sight comes into contact with the inner side of the tunnel.

The objection to the use of the theodolite in the case described was the

loss of time in adjusting the instrument laterally in prolongation of the wires, as at that time all lateral adjustment had to be made by moving the whole instrument and tripod, which is a tedious process. Since then several improvements have been made in theodolites for use in tunnel work, and they are now almost always provided with an arrangement which permits of a limited lateral movement of the horizontal plates over the legs, so that after the instrument has been set up the final adjustment may be made by means of the arrangement referred to. The objection mentioned above does not therefore hold good at the present time, and the theodolite method is most generally adopted on account of its superior accuracy.

To fix bench marks for levels at the bottom of the shafts, levels were taken at convenient places on the top of the shafts, and measurements were then made down the shafts by means of a steel tape. Levels can easily be taken underground by the use of lamps, one for shining on the staff, the other for flashing in front of the instrument to pick up the position of the hairs. For taking the levels in the headings, tripods with short legs were found most convenient, it being difficult to set up an instrument of the ordinary dimensions, so as to avoid obstacles in the line of sight.

It must not be thought that the example above described is in any way a difficult one, for as a matter of fact it is quite simple. It would occupy too much space to give a description of the methods adopted to set out a compound or reverse curve in a tunnel, when much greater difficulties are met with.

CHAPTER VI.

SPHERICAL TRIGONOMETRY.

Introductory Remarks.

'SPHERICAL trigonometry' might be called 'Solid trigonometry,' in contradistinction to 'Plane trigonometry,' which deals with the ratios which subsist between the sides and angles of a triangle, all of which are situated in one plane, viz. that of the paper on which the triangle is drawn. Spherical trigonometry, on the other hand, treats of the case in which the triangles are situated in three or more planes, more or less inclined to each other, which pass through some one point. Any three planes, passing through one point, and inclined to each other, enclose between them a pyramidal space. If more than three planes pass through the same point in space, then, as in the case of plane polygons, the enclosed pyramid may be subdivided into several pyramids, each being bounded by three planes only.

Any pyramid, enclosed by three planes which pass through a common point, is called a 'spherical triangle.' The object of 'spherical trigonometry' is to determine the plane trigonometrical ratios which obtain between the angles at which these three planes are inclined to each other (as measured in planes perpendicular to their 'lines of intersection,' or to the 'arrises' of the pyramid), as well as the angles which are included between the 'lines of intersection' or 'arrises' at the common point, through which the three planes pass.

It is usual, and convenient, to assume that the three intersecting planes are bounded by a 'spherical surface' of unit radius, having its centre at the common point of intersection. The three planes therefore, cut the bounding sphere in great circles. (A great circle is the intersection of the surface of a sphere with a plane passing through the centre. Its radius is, therefore, equal to that of the sphere, *vide* Part I.).

A 'spherical triangle' may, therefore, be represented in orthographic projection in the manner shown in fig. 37, in which one of the three planes is supposed to be situated in the plane of the paper.

A O B, A O C, and B O C, are the three planes, all passing through the common point O, of which the plane B O A coincides with the plane of the paper. The remaining planes B O C, A O C, are inclined to the plane of the paper, and are represented by their projections thereon. The figure B O A C is therefore a 'plan' of the pyramid, referred to the plane of one of its faces, namely B O A. The figure B A C represents the portion of the bounding sphere intercepted between the three planes, which form the faces of the pyramid. A model of the solid can be made by cutting a segment of a circle such as C_2 B A C_1 in paper,

and folding it together along the lines O B and O A, so that C_1 meets C_2 in C, and the lines O C_1 and O C_2 come together in the (projected) line O C. One has then, only to conceive that the pyramid is filled in with matter, and that its base A B C is finished off to a spherical surface, with O as its centre, to have a complete model of a spherical triangle.

Nomenclature. The arcs on the surface of the bounding sphere A B, B C, C A, are called the 'sides' of the spherical triangle, but these arcs are measured by the angles subtended between the lines A O and B O, B O and C O, A O and C O, respectively, at the centre of the sphere O.

The angles between the three planes, measured in planes perpendicular to

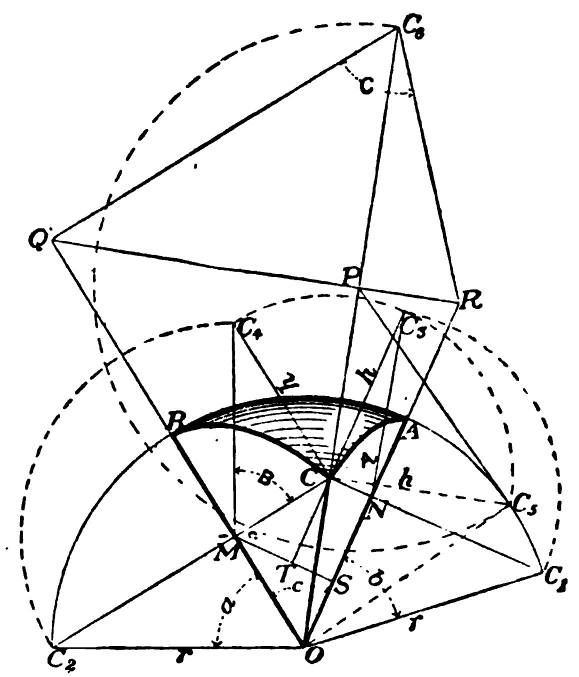


FIG. 37.

their 'lines of intersection,' are called the 'angles' of the spherical triangle. They are usually denoted by capital letters indicating the point at which any 'line of intersection' of two planes cuts the bounding sphere. The angle between the planes B O C and A O B (that of the paper) is called the angle B. The letter C denotes the angle between the planes B O C and A O C, whilst A denotes the angle between the two planes A O B and A O C. These angles are measured in planes perpendicular to the 'lines of intersection.'

It will, therefore, be seen that the 'spherical triangle' A B C (as regards geometry), exactly corresponds with the 'geodetic triangle,' measured on the earth's 'geodetic surface.' The angles A, B, and C, are the angles measured by the theodolite whose vertical axis is perpendicular to a tangential plane at each

point, and in the direction of the plumb-line, and moreover, the angular sides a, b, and c, are the geodetic distances from point to point *expressed in angles*, by dividing these distances in feet by the length of an arc of one degree, one minute, or one second, subtending these angles with the radius of the earth appropriate to the place.

The sides are usually indicated by small letters corresponding with the capitals indicating the angles.

The folding, and unfolding idea, leads to the simple graphical construction which solves the most common problems in 'spherical trigonometry,' and affords the means of establishing the fundamental trigonometrical ratios between the sides and angles, viz. :—

'Given the three sides (expressed in angular measure as above) of a spherical triangle, to find the three angles.'

With O as a centre and any convenient radius describe a circle. Then lay off successively the angles

$$\left.\begin{array}{l} C_2 O B = a \\ B O A = c \\ A O C_1 = b \end{array}\right\} \quad a,\ b,\ \text{and } c,\ \text{being the given sides.}$$

Imagine that the segment $C_2 B A C_1 O$ is cut out and folded together, the segments C_2 O B and C_1 O A revolving about the lines O B and O A until C_2 meets C_1 in C, and the radii O C_2 and O C_1 join in the line C O. The pyramid is now constructed. It is evident that the point C_2 sweeps through a circle, whose plane is perpendicular to B O, and the trace of which, on the plane of the paper, is the line C_2 M C, which is perpendicular to the radius O B. Similarly the line C_1 N C perpendicular to O A represents the trace of the plane in which C_1 moves when the plane C_1 O A revolves, in folding together. Lastly, the point C, where the two lines C_2 M C, C_1 N C intersect, is the perpendicular projection of the point C on the plane of the paper. Then, to measure the angles B and A we need only draw sections of the pyramid, in planes perpendicular to the paper, and to the lines O B, O A, respectively. Assume that these section planes are turned down into the plane of the paper by revolution about their traces, C M, C N, respectively.

Then, C_3 and C_4, the projections of C on the sectional planes, must be situated in lines C C_3 and C C_4 drawn through C, and perpendicular to C N and C M. Further, it is evident that the hypothenuse N C_3 and M C_4 must be equal to the perpendiculars C_2 M and C_1 N. Therefore, by drawing perpendiculars to C N and C M through C, and intersecting them with circles described with MC_2 and NC_3 as radii, we obtain two triangles C_3 N C and C_4 M C, whose angles at M and N are the angles which the planes C O B and A O C make with the plane of the paper, or in other words, the angles B and A of the spherical triangle A B C. The third angle C could be measured by supposing that the pyramid was placed with the plane B O C or A O C upon the plane of the paper, and proceeding as before. It can also be obtained, by direct construction, from the original diagram, as explained on page 81.

Proof of Sine Formula.

The above graphical solution leads at once to the fundamental trigonometrical relation between the sides and angles of a 'spherical triangle.'

(i.) Given two angles and one side, or two sides and one angle, to find the other opposite side or angle.

It is evident that $C\,C_3$ and $C\,C_4$ are equal, for both represent the perpendicular distance between C and the plane of the paper.

Then

$$\frac{C\,C_3}{N\,C_3} = \sin C_3\,N\,C = \sin A$$

and

$$\frac{C\,C_4}{M\,C_4} = \sin C_4\,M\,C = \sin B$$

But

$$M\,C_4 = C_2\,M = r \sin a$$

and

$$N\,C_3 = C_1\,N = r \sin b,$$

Substituting,

$$\frac{C\,C_4}{r \sin a} = \sin B$$

or

$$C\,C_4 = r \sin B \sin a$$

and similarly

$$C\,C_3 = r \sin A \sin b$$

Therefore, as

$$C\,C_3 = C\,C_4$$

$$r \sin B \sin a = r \sin A \sin b$$

or

$$\frac{\sin B}{\sin b} = \frac{\sin A}{\sin a}$$

A similar ratio may be obtained for the side c and angle C, and we may say—

$$\sin A : \sin B : \sin C :: \sin a : \sin b : \sin c \quad . \quad . \quad \text{(I)}$$

That is to say, 'the sines of the sides of a spherical triangle are proportional to the sines of the angles to which they are opposite.'

Other Formulæ.

Other fundamental formulæ relating to spherical triangles can be deduced from the same figure, as follows.

(ii.) Given three sides, to find the values of the angles—

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

Similarly,

$$\cos B = \frac{\cos b - \cos c \cos a}{\sin c \sin a} \quad . \quad . \quad . \quad \text{(II)}$$

(iii.) Given the three angles, to find the values of the sides—

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \quad . \quad . \quad . \quad \text{(III)}$$

(iv.) Given two sides and the included angle, or two angles and the included side, to find the remaining functions—

$$\cot A \sin B = \cot a \sin c - \cos B \cos c \quad . \quad . \quad \text{(IV)}$$

To prove formula (II) viz.

$$\cos B = \frac{\cos b - \cos c \,.\, \cos a}{\sin c \,.\, \sin a}$$

Draw M S (fig. 37) parallel to $C C_1$, and C T parallel to O A.

Then

$$\text{angle } O S M = \text{angle } C N O = 90°$$

$$\therefore \ \text{angle } S M O + c = 90°$$

But

$$\text{angle } S M O + \text{angle } T M C = 90°$$

$$\therefore \ T M C = c$$

Now

$$\cos b = \frac{O N}{O C_1}$$

$$= \frac{O S}{O C_1} + \frac{S N}{O C_1}$$

$$= \frac{O S}{O M} \cdot \frac{O M}{O C_1} + \frac{S N}{M C} \cdot \frac{M C}{M C_4} \cdot \frac{M C_4}{O C_1}$$

$$= \cos c \,.\, \frac{O M}{O C_2} + \frac{T C}{M C} \,.\, \cos B \,.\, \frac{M C}{O C_2}$$

$$= \cos c \,.\, \cos a + \sin c \,.\, \cos B \,.\, \sin a$$

whence

$$\cos B = \frac{\cos b - \cos c \,.\, \cos a}{\sin c \,.\, \sin a}$$

To prove formula (III) viz.

$$\cos c = \frac{\cos C + \cos A \,.\, \cos B}{\sin A \,.\, \sin B}$$

Find the angle C as on page 81 = $Q C_6 R$.

Then $$Q R^2 = C_6 R^2 + C_6 Q^2 - 2 C_6 R \,.\, C_6 Q \cos C$$
$$= O R^2 + O Q^2 - 2 O R \,.\, O Q \cos c$$

$$\therefore - 2 C_6 R \,.\, C_6 Q \cos C$$

$$= O R^2 - C_6 R^2 + O Q^2 - C_6 Q^2 - 2 O R \,.\, O Q \cos c. \quad . \quad (a)$$

Now

$C_6 R = C_1 R$, since each of them gives the development of C R.

Similarly,

$$C_6 Q = C_2 Q, \text{ whence } C_6 Q = O C_2 \tan a = r \tan a$$
$$\text{and } C_6 R = O C_1 \tan b = r \tan b$$

$$\therefore \quad 2 r^2 . \tan a \tan b . \cos C = - 2 r^2 + 2 r^2 \sec a \sec b \cos c, \text{ from } (a)$$

$$\therefore \quad \cos C . \tan a \tan b = - 1 + \sec a \sec b \cos c$$

or

$$\cos C = - \frac{1}{\tan a \tan b} + \frac{\cos c}{\sin a \sin b} \quad . \quad . \quad . \quad . \quad (b)$$

Now

$$\sin a \sin B = \sin b \sin A = \frac{h}{r}, \text{ where } h = C C_4 = C C_3$$

$$\therefore \quad \frac{r^2 \sin A \sin B}{h^2} = \frac{1}{\sin a \sin b}$$

$$\therefore \quad \cos C = \cos c \frac{r^2}{h^2} \sin A \sin B - \frac{O M}{M C_4} . \frac{O N}{N C_3}$$

$$= \cos c . \sin A \sin B + \frac{r^2 - h^2}{h^2} \cos c \sin A \sin B - \cos B \cos A \frac{O M}{C M} . \frac{O N}{C N}$$

$$= \cos c . \sin A \sin B - \cos B \cos A + \frac{r^2 - h^2}{h^2} \cos c \sin A \sin B$$

$$- \cos B \cos A \left(\frac{O M . O N}{C M . C N} - 1\right)$$

Now

$$\frac{r^2 - h^2}{h^2} . \cos c \sin A \sin B - \cos B \cos A \left(\frac{O M . ON - C M . C N}{C M . C N}\right)$$

$$= \frac{O M^2 + M C^2}{h^2} . \cos c \ \frac{h^2}{M C_4 . N C_3} - \frac{C M . C N}{M C_4 . N C_3}\left(\frac{O M . O N - C M . C N}{C M . C N}\right)$$

$$= \frac{O M . O S + M C . M T - O M . O N + C M . C N}{M C_4 . N C_3}$$

$$= \frac{M C . M S - O M . S N}{M C_4 . N C_3} = \frac{\operatorname{cosec} c (C T . M S - M S . S N)}{M C_4 . N C_3} = 0$$

$$\therefore \quad \cos C = \cos c \sin A \sin B - \cos A \cos B$$

or

$$\cos c = \frac{\cos C + \cos A . \cos B}{\sin A . \sin B}$$

To prove formula (IV) viz.

$$\cot A \sin B = \cot a \sin c - \cos B \cos c$$

$$\cot A \sin B = \frac{N C}{C C_3} . \frac{C C_4}{M C_4}$$

$$= \frac{N C}{M C_4} \text{ (since } C C_4 = C C_3)$$

$$= \frac{SM}{MC_4} - \frac{TM}{MC_4}$$

$$= \frac{SM}{OM} \cdot \frac{OM}{MC_4} - \frac{TM}{MC} \cdot \frac{MC}{MC_4}$$

$$= \sin c \,.\, \frac{OM}{MC_2} - \cos c \,.\, \cos B$$

$$= \sin c \,.\, \cot a - \cos c \,.\, \cos B$$

To find the third angle at C, construct the projection of the pyramid A B C O as before.

The angle C is measured in a plane perpendicular to the edge of the pyramid represented by its projection O C_5. Take therefore a section of the pyramid by a plane perpendicular to the paper, passing through C and O. Assume this sectional plane to be turned down to the right. Then it is clear that C_5 the projection of C on the plane turned down must be situate in a line perpendicular to O C, drawn through C. It is also obvious that the point C_5 must be the intersection of the said perpendicular with the circle C_2 B A C_1, because C_5 O represents the true length of the edge C O, or the radius of the bounding sphere.

Now the plane in which the angle C is to be measured is perpendicular to the line C O whose projection on the sectional plane is O C. The trace of the measuring plane on the sectional is the line C_5 P perpendicular to the line C_5 O. This line cuts C O prolonged in P, through which point the trace of the measuring plane, on the plane of the paper, must pass.

The measuring plane being perpendicular to the edge C O, its horizontal trace must be perpendicular to the horizontal projection of C O. Now, draw the line Q P R through P perpendicular to O P, and produce O B and O A to cut it in Q and R. Join C Q and C R. Then, the triangle C Q R is the projection of a fourth (plane) face of the pyramid whose plane is perpendicular to the edge C O, or in other words, tangent to the bounding sphere at C.

To obtain the true measure of the angle C, imagine that the triangle C Q R is turned down to the plane of the paper, by revolving about its horizontal trace Q P R. Then C_6, the projection of C on the plane of the paper, will be situated on O C produced, the distance P C_6 being obviously equal to P C_5, so that the angle Q C_6 R is the angle C.

Logarithmic Adaptations. The above formulæ, with the exception of the first, are not adapted for logarithmic computation. They may however, be readily transformed into others which are, by methods precisely analogous to those adopted in plane trigonometry.

Thus,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$\therefore \quad 1 - \cos A = \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c}$$

$$= \frac{2 \sin \left(\frac{b - c + a}{2}\right) \sin \left(\frac{a - b + c}{2}\right)}{\sin b \,.\, \sin c}$$

$$\therefore \quad 2 \sin^2 \frac{A}{2} = \frac{2 \sin (s - c) \,.\, \sin (s - b)}{\sin b \sin c}$$

or

$$\left.\begin{aligned} &\sin^2 \frac{A}{2} = \frac{\sin (s - c) \sin (s - b)}{\sin b \sin c}, \text{ where } s = \frac{a + b + c}{2} \\ &\text{Similarly,} \\ &\cos^2 \frac{A}{2} = \frac{\sin s \sin (s - a)}{\sin b \sin c} \\ &\text{and} \\ &\tan^2 \frac{A}{2} = \frac{\sin (s - c) \sin (s - b)}{\sin s \sin (s - a)} \end{aligned}\right\} \quad \text{(V)}$$

Similar formulæ hold good for the other angles, and enable us to find any angle when the '*three sides*' are known.

Again,

$$\tan \tfrac{1}{2} (A + B) = \frac{\tan \frac{1}{2} A + \tan \frac{1}{2} B}{1 - \tan \frac{1}{2} A \,.\, \tan \frac{1}{2} B}$$

Substituting for $\tan \frac{1}{2} A$ and $\tan \frac{1}{2} B$ by the last formulæ and reducing, this becomes

$$\tan \tfrac{1}{2} (A + B) = \left(\frac{\sin (s - c) \,.\, \sin s}{\sin (s - a) \sin (s - b)}\right)^{\frac{1}{2}} . \left(\frac{\sin (s - b) + \sin (s - a)}{\sin s - \sin (s - c)}\right)$$

$$= \frac{2 \sin \frac{2s - b - a}{2} \,.\, \cos \left(\frac{a - b}{2}\right)}{2 \sin \frac{c}{2} \cos \frac{2s - c}{2}} \cot \tfrac{1}{2} C$$

$$= \frac{\sin \frac{c}{2} \,.\, \cos \frac{a - b}{2}}{\sin \frac{c}{2} \,.\, \cos \frac{a + b}{2}} \cot \tfrac{1}{2} C$$

$$\left.\begin{aligned} &= \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)} \cot \tfrac{1}{2} C \\ &\text{Similarly,} \\ &\tan \tfrac{1}{2} (A - B) = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)} \cot \tfrac{1}{2} C \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(VI)}$$

Similar formulæ hold for the other angles, and they enable us to find the remaining angles when '*two sides and the included angle*' are known.

Again, starting with the formula—

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

and working in a similar way, we obtain

$$\left.\begin{aligned} \sin^2\frac{a}{2} &= \frac{-\cos S \,.\, \cos(S - A)}{\sin B \sin C} \\ \cos^2\frac{a}{2} &= \frac{\cos(S - C) \,.\, \cos(S - B)}{\sin B \sin C} \\ \text{and} \qquad \tan^2\frac{a}{2} &= -\frac{\cos S \,.\, \cos(S - A)}{\cos(S - C) \,.\, \cos(S - B} \end{aligned}\right\} \quad . \quad \text{(VII)}$$

where $S = \frac{1}{2}(A + B + C)$.

Similar formulæ are true for the other sides, and enable us to find the sides when the '*three angles*' are known.

Lastly, working from these, as before, we get

$$\left.\begin{aligned} \tan\tfrac{1}{2}(a + b) &= \frac{\cos\frac{1}{2}(A - B)}{\cos\frac{1}{2}(A + B)}\tan\tfrac{1}{2}c \\ \text{and} \qquad \tan\tfrac{1}{2}(a - b) &= \frac{\sin\frac{1}{2}(A - B)}{\sin\frac{1}{2}(A + B)}\tan\tfrac{1}{2}c \end{aligned}\right\} \quad . \quad \text{(VIII)}$$

formulæ which enable us to find the remaining sides when '*one side and the angles on each side of it*' are known.

Right-Angled Triangles.

If in all these equations we put $C = 90°$, we obtain a new set of equations which apply to right-angled spherical triangles.

From (I)

$$\sin a = \sin c \,.\, \sin A \quad . \quad . \quad . \quad . \quad (1)$$

similarly

$$\sin b = \sin c \,.\, \sin B \quad . \quad . \quad . \quad . \quad (2)$$

From (II)

$$\cos c = \cos a \cos b \quad . \quad . \quad . \quad . \quad (3)$$

From (III)

$$\cos A = \cos a \,.\, \sin B \quad . \quad . \quad . \quad . \quad (4)$$

similarly

$$\cos B = \cos b \,.\, \sin A \quad . \quad . \quad . \quad . \quad (5)$$

From (IV)

$$\cot A = \cot a \sin b \quad . \quad . \quad . \quad . \quad (6)$$

or

$$\tan a = \tan A \sin b \quad . \quad . \quad . \quad . \quad (7)$$

similarly

$$\tan b = \tan B \sin a \quad . \quad . \quad . \quad . \quad (8)$$

or

$$\cot c \sin a = \cos B \cos a$$

or

$$\tan a = \tan c \,.\, \cos B \qquad . \quad . \quad . \quad (9)$$

similarly

$$\tan b = \tan c \,.\, \cos A \qquad . \quad . \quad . \quad (10)$$

All cases of right-angled triangles may be solved by means of these formulæ.

Adaptation of Spherical Trig. Formulæ to the requirements of Sph. Astronomy.

It remains to select the formulæ of 'spherical trigonometry' found most useful in solving problems in 'spherical astronomy,' and to adapt them to the nomenclature of the 'astronomical triangle.'

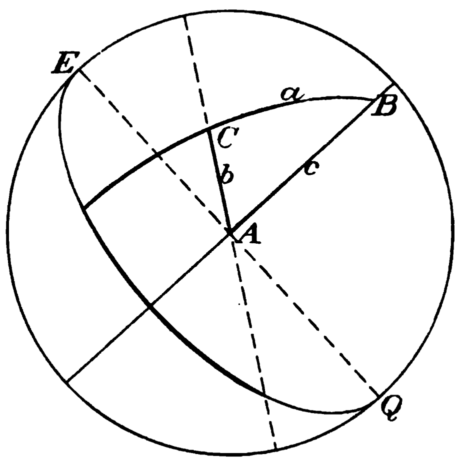


FIG. 38.

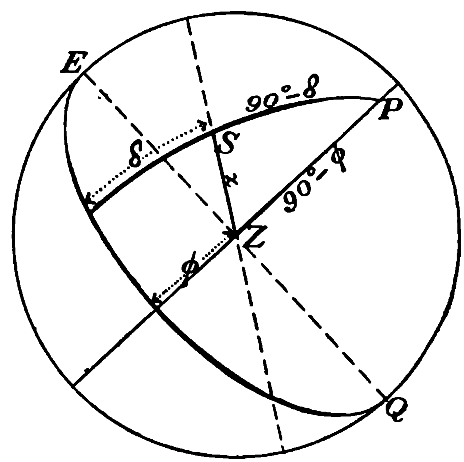


FIG. 39.

The required change in nomenclature is indicated in the above similar spherical triangles (figs. 38 and 39). One A B C, is used in spherical trigonometry, and the other P Z S is used in astronomical calculations. The astronomical triangle is drawn in plan on the horizon plane. The equivalent angles and sides are as follows.

Angle B	=	S P Z	=	t = hour angle.	(*Vide* Chap. VIII.)
„ A	=	P Z S	=	supplement of azimuth.	
„ C	=	P S Z	=	the parallactic angle.	
Side b	=	S Z	=	z = zenith distance.	
„ a	=	S P	=	$90 - \delta$ = co-declination.	
„ c	=	P Z	=	$90 - \phi$ = co-latitude.	

Formulæ for Calculating Azimuth and Time.

When calculating 'azimuth' and 'time,' the following formulæ for a spherical triangle are used, and their adaptation to the astronomical triangle is also given.

$$\tan^2\frac{A}{2} = \frac{\sin(s-c)\sin(s-b)}{\sin s\sin(s-a)}$$

$$= \frac{\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}}{\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}} \quad \ldots \quad (a)$$

$$\tan^2\frac{B}{2} = \frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}$$

$$= \frac{\sin\frac{b+c-a}{2}\sin\frac{a+b-c}{2}}{\sin\frac{a+b+c}{2}\sin\frac{a+c-b}{2}} \quad \ldots \quad (b)$$

Now

$$\angle A = \angle PZS = \text{azimuth from elevated pole.}$$

and

$$\angle B = \angle SPZ = \text{hour angle } t.$$

Also

a = PS = $90 - \delta$ = co-declination or polar distance = p.
b = ZS = $90 - h$ = zenith distance = z.
c = PZ = $90 - \phi$ = co-latitude.

Hence, if s be used to indicate the semi-sum $\frac{h+\phi+p}{2}$

$$\frac{a+b-c}{2} = \frac{p+\phi-h}{2} = s-h$$

$$\frac{a+c-b}{2} = \frac{p+h-\phi}{2} = s-\phi$$

$$\frac{b+c-a}{2} = 90 - \frac{p+\phi+h}{2} = 90 - s$$

$$\frac{a+b+c}{2} = 90 - \frac{h+\phi-p}{2} = 90 - (s-p)$$

And substituting these values in (a) and (b), we have

I. For azimuth,

$$\tan^2\frac{A}{2} = \sin(s-h)\sin(s-\phi)\sec(s-p)\sec s$$

II. For time,

$$\tan^2\frac{t}{2} = \sin(s-h)\sec(s-p)\operatorname{cosec}(s-\phi)\cos s$$

CHAPTER VII.

THE FIGURE OF THE EARTH.

The Earth's Geodetic Surface.

When we speak of the 'earth' as a sphere or spheroid, we do not mean that the external or visible surface of the earth is such, but that the *mean* 'surface of the sea' (produced in imagination so as to percolate the continents), forms a regular surface of revolution. The reduction of 'observed distances' to 'geodetic distances' has already been treated of in connection with the trigonometrical determination of 'vertical heights.' A 'surface of equal altitude' has been defined in the Chapter on 'Levelling.' Following the simile adopted in that case, the 'earth's surface' as referred to in 'Geodesy,' may be realised by imagining that the continents are traversed in every direction by subterranean channels, below sea level, to which the sea has complete access, and further, that numerous wells or shafts are sunk and connected to this underground network of channels. Then the mean level of the water in the wells would coincide with the 'geodetic surface' of the 'earth.' This 'surface' is that to which all 'geodetic measurements' are referred (*vide* Part I., p. 252).

'Levels,' or 'altitudes' of terrestrial places, are the vertical distances between the 'geodetic surface,' and the actual surface of the ground. 'Geodetic distances' are measured on the 'geodetic surface.'

General form of Geodetic Surface.

The general form of the earth, as bounded by the 'geodetic surface' as above defined, is an oblate spheroid, that is to say, the solid generated by the revolution of an ellipse, about its shorter or minor axis. The difference in length of the two axes of this ellipse is small, the ratio being approximately as 300 is to 301. For many purposes therefore the 'geodetic surface' may be treated as a sphere.

Effect on the Direction of the Force of Gravity.

One important property of the 'geodetic surface,' is, that a 'plumb-line,' suspended above any point in it, will be normal to that 'surface,' that is to say, the plumb-line will be perpendicular to a plane, tangential to the surface, at the point immediately below the 'plumb-line.'

Now the 'plumb-line' indicates the direction of the resultant of the force of gravity, that is to say, the direction of the resultant of the attractions of the particles of matter, which form the mass of the earth, combined with the centrifugal force, due to the earth's rotation, and it is essential to the stability of the figure of the earth, that this should be normal to the 'geodetic surface.' Were the resultant of the various attractions in a direction other than normal, then the form of the 'geodetic surface' would be unstable, and change of shape would forthwith take place.

The Polar Axis. The shorter axis of the ellipse, which by its revolution, generates the ellipsoid that coincides with the geodetic 'surface' of the 'earth,' is the 'polar axis.' This is the axis about which the earth rotates daily. The direction and position of the 'polar axis' is determined by astronomical observation. As a matter of common observation, the fixed stars appear to revolve round the earth in circles, the planes of which are parallel to each other. If the apparent movement of the stars be carefully observed, it will be seen that the circles in which they revolve, become smaller, as a certain point in the heavens is approached. A star, situate near to this point, will not appear to move at all. This point is called the 'celestial pole,' and marks the prolongation of the axis, about which the earth revolves. A line drawn through the centre of the earth to the 'celestial pole' cuts the 'earth's surface' in the 'terrestial poles,' which are called 'north' and 'south.' A plane, passing through the centre of the earth and perpendicular to the 'polar axis,' is called an 'equatorial plane,' and the great circle in which it cuts the 'earth's surface' is called the 'equator.'

Geocentric Latitude. The angle which a line from the 'centre of the earth,' to any point in its surface makes with the 'plane of the equator' is called the 'geocentric latitude' of that point. Were the earth a true sphere, 'geocentric latitude' would serve as a definition of 'latitude' generally, but such is not the case.

Geographical Latitude Defined. The 'geographical latitude' of a point on the 'earth's surface' may be defined as the 'angle' which a 'plumb-line,' suspended over the point in question, produced, makes with the 'plane of the equator.'

Thus, referring to fig. 40, let C be the centre of the earth, N C S its axis, and E Q the equator, whose plane is at right angles to the axis.

Let A be the point on the earth's surface, whose latitude is required, and let Z A or Z_1 A be the direction of the plumb-line at A.

If the earth were a true sphere, then Z A prolonged, would pass through C, the centre. If it be not a true sphere, but an ellipsoid, then the line Z_1 A, representing the direction of the plumb-line, would pass through some other point C_1 in the plane of the equator. In either case, the angle A C E, or A C_1 E as the case may be, is the latitude of the point A. The angle A C E is termed the 'geocentric latitude,' whilst A C_1 E is termed the 'geographical latitude.'

This latter angle is determined, astronomically, on the principles here indicated, as follows:—

Suppose that there were a fixed star P, exactly in the prolongation of the earth's polar axis, and therefore marking the celestial pole. At A, let the angle P_1 A Z_1 between the star P and the plumb-line be measured. Since the distance of the star is indefinitely great compared with the radius of the earth, the angle P_1 A Z_1 is equal to the angle P_2 C_1 Z_1 which is the complement of the angle A C_1 E, the 'geographical latitude' of the place. Hence, the latitude of any place may be defined as '*the complement of the angle between the plumb-line and the celestial pole.*'

Again, let *n* A *s* be the projection of a plane, touching the earth at A. This plane is called the plane of the Sensible Horizon, and the plumb-line is perpen-

dicular to it. The line of collimation of a correctly adjusted level, set up at A, is exactly parallel to the plane of the Sensible Horizon in every direction. Now let the angle $P_1 A n$ be observed (which is called the altitude of the celestial pole), then as $P_1 A$ is parallel to $P_2 C_1$, it is evident that the angle $P_1 A n$ is equal to $A C_1 E$ the geographical latitude of A, and therefore the latitude of a place is equal to '*the altitude of the celestial pole as observed at that place.*'

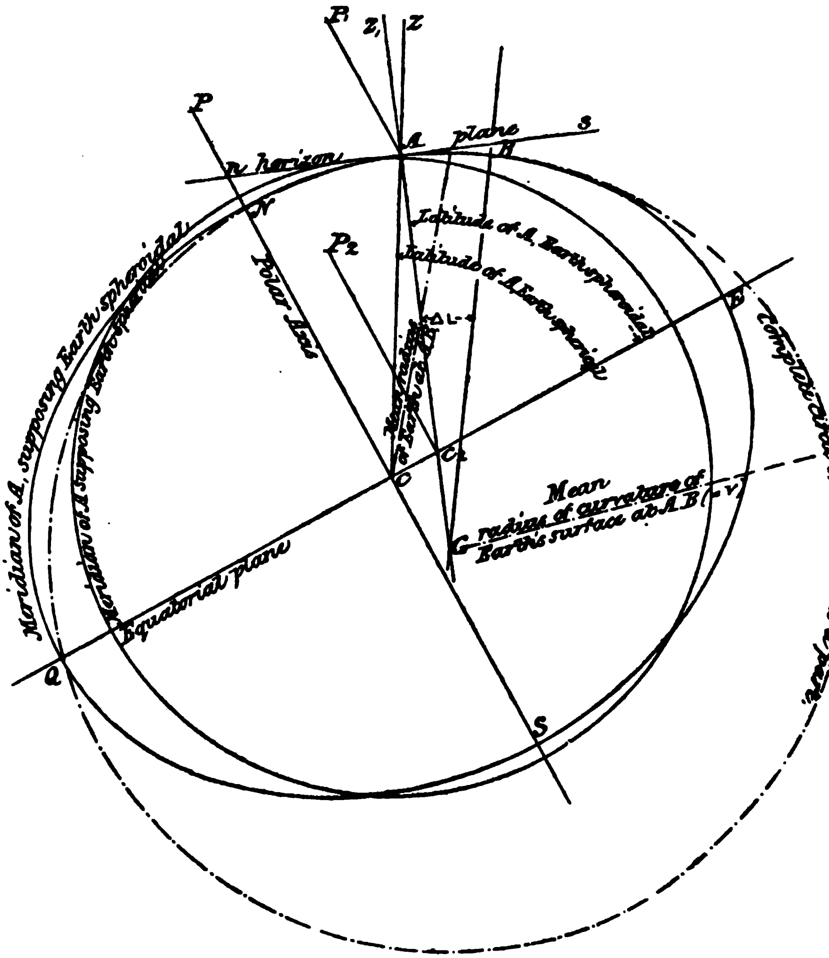


FIG. 40.

How the position of the celestial pole is fixed, in the absence of a star actually situated in it, will be explained in the chapter on astronomical observations.

Determination of the 'Radius of Curvature' of the Arc A B.

Now suppose that observations for latitude were made at two points A and B not very far (but due north and south) from each other, then the 'difference of latitude' between A and B is known. Let Δ L be this 'difference of latitude,' and let D be the 'geodetic distance' between the two points, measured

by triangulation. Then D is the length of an arc of a circle (really part of an ellipse) passing through A and B, and which subtends an angle Δ L at the centre of the circle (or ellipse). Let r be the radius of the circle, coinciding, locally with the arc of the ellipse. Then

$$2\pi r = \text{the whole circumference of the circle}$$

and
$$\frac{2\pi r \times \Delta L}{360^\circ} = D$$

$\therefore$
$$r = \frac{D \times 360^\circ}{2\pi \times \Delta L}$$

where Δ L = difference of latitude between A and B in degrees.

The radius of the earth for the small arc Δ L is therefore determined, or in other words the 'radius of curvature' of the 'meridian' for the 'mean latitude' of the arc A B. The 'radius of curvature' of an arc of the earth's surface must not be confounded with the 'radius of the earth,' that is to say, the distance from the earth's centre, to a point in its surface, at the middle of such arc. The length of an 'arc of the meridian,' subtending a 'unit angle' at any point, is obviously $\frac{D}{\Delta L}$. Now, were the lengths of such 'arcs of the meridian,' as determined at different places (some near to the 'equator,' others removed from it), found to be in every case exactly equal, then the earth would be a 'true sphere.' Such, however, is not the case, but it is found that the length of an 'arc of a meridian' subtending say 1°, is less when measured near the 'equator' than when approaching the 'poles,' that is to say, the radius of the curvature of the 'meridian' is greater at the 'poles' than at the 'equator.' This leads to the conclusion that the form of the earth is an oblate spheroid, and that the section of the earth, through the poles, is approximately an ellipse.*

This is made clear in fig. 41. The ellipticity is much exaggerated, but the geometry is not thereby altered. The full lines are normal to the surface and represent successive plumb-lines, inclined at 10° to one another. It will be seen that as the pole is approached, the radii of curvature 'become longer,' the centres of curvature recede from the plane of the equator, and the arc opposite the angle of 10°, gradually gets longer in consequence. The broken lines show geocentric radii, also inclined at 10°. It will be noticed that these are shortest at the poles. Hence, if they gave the directions of the plumb-line, an arc of 1° would be shortest at the poles also.

Longitude Defined. The longitude of a place is the arc of the equator, intercepted between the meridian of the place, and the meridian of some standard place such as Greenwich, from which longitude is measured east or west up to 180° or 12 hours in time.

* The fact that the form of the Earth is an Oblate Spheroid, as stated on page 86, was originally demonstrated, in 1736–37, by Maupertius and La Condamine, who measured respectively the length of an arc of a degree of the meridian in Lapland and Peru, respectively. Since then many arcs of the meridian have been measured, from which the ellipticity of the earth has been determined, with close accuracy.

Let A and G, fig. 42, be two points on the earth's surface, let planes N A Q, N G_1 G G_2 be drawn through the earth's polar axis and the two points in ques-

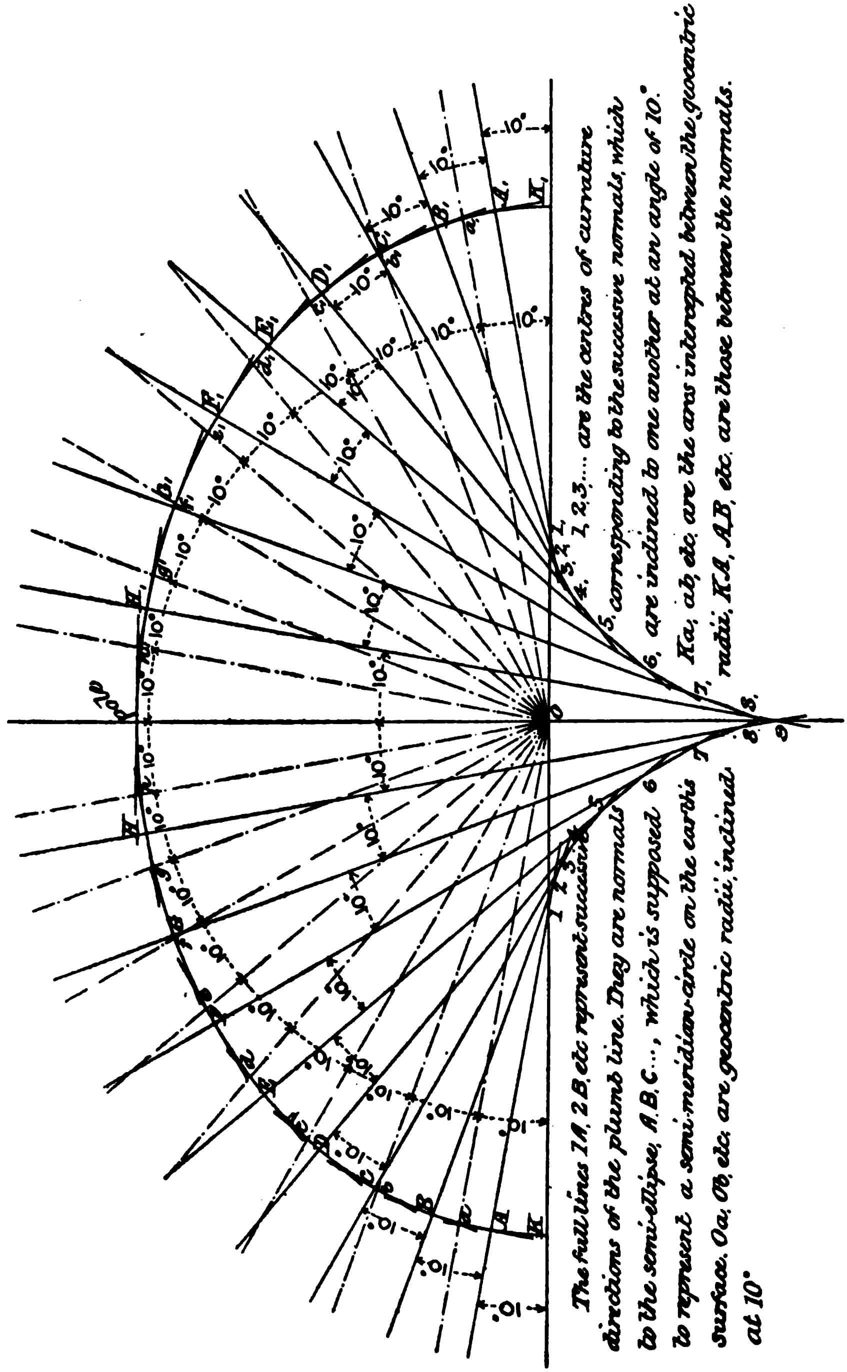


FIG. 41.

tion, then the spherical angle contained between these planes, and measured in a plane at right angles to the axis, or parallel to the plane of the equator, is the difference of longitude between the two points. Thus, the angles $A\,N_2\,G_1$, $A_1\,N_1\,G$, or $Q\,C\,G_2$ are the difference of longitude between A and G. The longitudes of places on the earth's surface are measured from some one standard meridian plane. That passing through the Observatory of Greenwich is usually adopted as the 'zero plane' by the English, whilst the French use that of Paris, the Germans that of Berlin, and so on. The angles of longitude are usually (though most inconveniently), reckoned 'east' and 'west' from the standard 'meridian,'

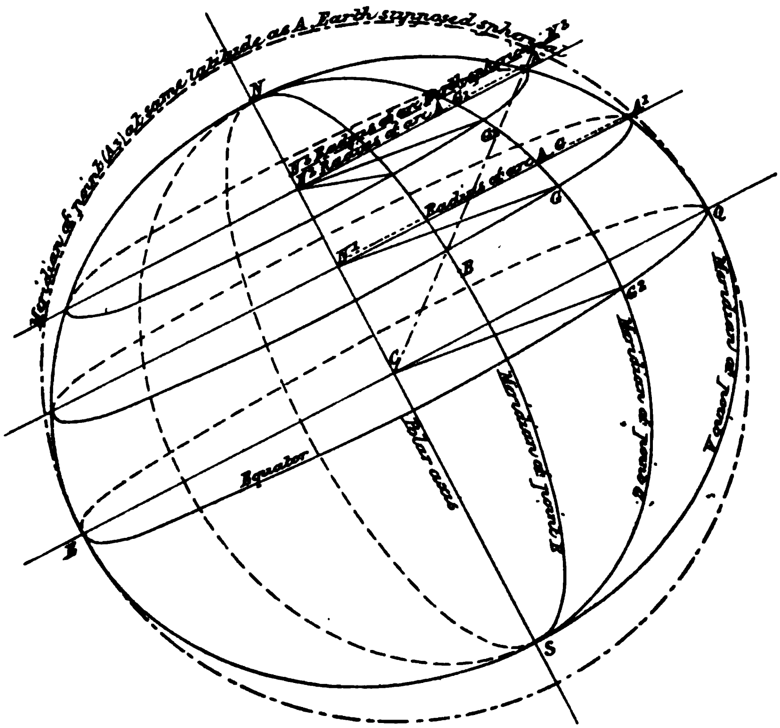


FIG. 42.

numbering from 0° to 180°. Thus the point A is, say, $n°$ east longitude, assuming that G is the Greenwich observatory, the longitude of the point B is then $n°$ west longitude.

Inasmuch as time is so generally used in determining differences of longitude, the latter are often expressed in time, that is to say, the point is said to be so many hours, minutes, and seconds, 'east' or 'west' of Greenwich. This means two things, firstly, the actual time interval elapsing between the passage of a given star over two different 'meridians' (the time being reckoned by means of a clock so regulated as to indicate exactly 24 hours = 1440 minutes = 86,400 seconds, during the interval of time which elapses between two

successive transits of a star over the same 'meridian,' or the time which the earth takes to make one complete revolution about its axis), and secondly, the trigonometrical angle between the two 'meridian planes,' that is to say, the fraction of four right angles, intercepted between the two planes, only that in this case four right angles form a whole turn, and are divided into twenty-four parts or hours, being further subdivided into 1440 minutes and 86,400 seconds, instead of being divided, as is more commonly the case in geometry, into 360 parts called degrees, and further sub-divided into 21,600 minutes of arc, and 1,296,000 seconds of arc. These two methods of denoting the sub-divisions of complete circles give rise to much inconvenience. They are, however, very easily convertible, being measured by a geometrical angle or arc in terms of a complete turn, or the fraction of the circumference of a circle. Thus,

$$\text{One hour} = \frac{360^\circ}{24} = 15^\circ \text{ of arc.}$$

$$\text{And one minute} = \frac{360 \times 60'}{24 \times 60} = 15' \text{ of arc.}$$

$$\text{And one second} = \frac{360 \times 60 \times 60''}{24 \times 60 \times 60} = 15'' \text{ of arc.}$$

$$\text{Conversely, } 1^\circ \text{ of arc} = \frac{24 \times 60'}{360} = 4 \text{ minutes of time.}$$

$$1' \text{ of arc} = \frac{24 \times 60 \times 60''}{360 \times 60} = 4 \text{ seconds of time.}$$

$$1'' \text{ of arc} = \frac{24 \times 60 \times 60}{360 \times 60 \times 60} = \tfrac{1}{15} \text{ second of time.}$$

Comparisons of the Lengths of Arcs of Longitude to indicate the Spheroidal shape of the Earth.

The measurements of the lengths of 'arcs of longitude' in different 'parallels of latitude' (that is to say, along the circumference of the circles in which planes parallel to the equator cut the earth's surface) substantiate the fact of the spheroidal form of the earth deduced from measurements made on a 'meridian.' If the earth were a sphere, then the lengths of 'arcs of longitude' measured on 'parallels of latitude' would be the lengths of such 'arcs of longitude' measured at the 'equator' multiplied by the cosines of the respective 'latitudes.' This is not the case, however for the lengths of 'arcs of longitude' are found to *decrease*, as the 'poles' are approached, at a more rapid rate than the cosine of the 'latitude.'

Let E N Q, fig. 43, represent the elliptical section of the earth on the 'plane of the meridian' of the point A. Again, let any section of the spheroid be made by a plane passing through A and C at right angles to the planes of the meridian. The line A C is the *trace* of this plane, and the axes of the elliptic section may be obtained by laying off CC_2 at right angles to A C and making CC^2 equal to C Q the radius of the equator. It is clear that the two sections differ considerably in

their ellipticity, since A C is greater than N C, and their radii of curvature at the common point A differ also.

O and O_1 are the two centres of curvature. The difference of the two 'radii of curvature' is greatest at the equator, and there, the 'radius of curvature' in the plane at right angles to the meridian, which is the equator itself, is the longest radius of the earth.

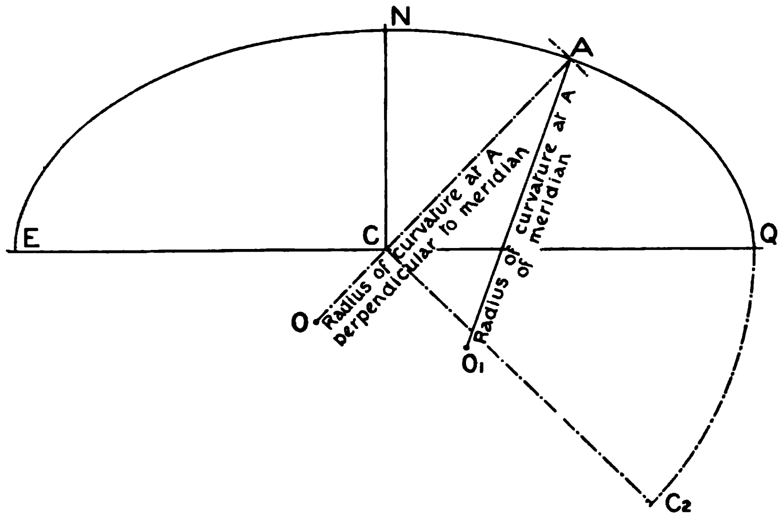


FIG. 43.

Dimensions of the Earth.

According to the latest determination, as published by Captain Clarke in the *Philosophical Magazine* for August, 1878 ('Units and Physical Constants,' by Professor Everett, Chapter LXV.), the semi-axes of the ellipsoid, which most nearly agree with the actual measurements of the earth are, in feet, as follows :—

Greatest semi-axis major, or greatest semi-diameter at equator	=	20,926,629
Least ditto	=	20,925,105
Mean ditto	=	20,925,867
Semi-axis minor, or semi-polar diameter	=	20,854,477
Mean radius	=	20,890,172

This mean radius differs from that given on page 167, Part I., namely 20,889,000, by 1172 feet only, or less than a quarter of a mile. The difference is due to the more accurate measurements which have become available, since the publication of Rankin's 'Treatise on Civil Engineering,' from which the dimensions, given in Part I., were taken.

The ellipticity of the two 'principal meridians,' is as follows:—

(1) That passing through the longest diameter of the equator $= \frac{1}{289 \cdot 0}$

(2) That passing through the shortest diameter $= \frac{1}{295 \cdot 3}$

whilst the ellipticity of the equator $= \frac{1}{13730}$ (a negligible quantity)

The following formulæ express the radii of curvature of the earth's surface, at any point, in different directions, neglecting the ellipticity of the equator, and taking the section on any meridian as an ellipse:—

(1) On the meridian

$$\rho = (1 + 2e - 3e\cos^2 L) \times b$$

(2) On the plane perpendicular to the meridian (the prime vertical)

$$\rho_{,} = (1 + 2e - e\cos^2 L) \times b$$

(3) On any other plane, passing through the centre of the earth and making an angle θ, with the plane of the meridian,

$$\rho_{,,} = \frac{\rho\,\rho_{,}}{\rho\sin^2\theta + \rho_{,}\cos^2\theta}.$$

where the meanings of the different letters are—

ρ = radius of curvature in the meridian proper to the point in question.
$\rho_{,}$ = " " prime vertical (perpendicular to meridian).
$\rho_{,,}$ = " " plane inclined θ° to meridian.
e = ellipticity of the meridian.

$$= \frac{\text{semi-axis major} - \text{semi-axis minor}}{\text{semi-axis minor}}$$

L = latitude of a point on earth's surface where the radius of curvature is required.

b = semi-axis minor, or polar radius = 20,854,477 feet.

The ellipticity of the equator being neglected, the *mean* semi-axis major would be taken in calculating the value of e. The result will thus be the mean of the two values of e given above, that is $\frac{1}{2}\left(\frac{1}{289} + \frac{1}{295 \cdot 3}\right)$.

Hence $$e = \frac{1}{292 \cdot 1}$$

From these formulæ we can calculate the lengths of the arc subtending an angle of 1′, at a point whose latitude is L, in each of these directions, as follows.

$$\text{Arc of } 1' = \frac{\text{whole circumference}}{360 \times 60}$$

$$= \frac{2\pi \times \text{radius of curvature}}{360 \times 60}$$

$$= \frac{\pi \times \text{radius of curvature}}{180 \times 60}$$

Now on the meridian

$$\rho = b(1 + 2e - 3e\cos^2 L)$$

$$= b\left\{1 + 2e - \frac{3e}{2}(1 + \cos 2L)\right\} \text{ since } \cos^2 L = \frac{1}{2}(1 + \cos 2L)$$

$$= b\left(1 + \frac{e}{2} - \frac{3e}{2}\cos 2L\right)$$

$$\therefore \quad \text{arc of } 1' = \frac{\pi\rho}{180 \times 60} = \frac{\pi b}{180 \times 60}\left(1 + \frac{e}{2} - \frac{3e}{2}\cos 2L\right)$$

$$= \frac{\pi b}{180 \times 60}\left(1 + \frac{1}{584 \cdot 2} - \frac{3}{584 \cdot 2}\cos 2L\right)$$

$$= \frac{\pi b}{180 \times 60 \times 584 \cdot 2}(585 \cdot 2 - 3\cos 2L)$$

$$= 10 \cdot 384\,(585 \cdot 2 - 3\cos 2L)$$

$$= 6077 - 31\cos 2L \text{ feet, nearly} \quad . \quad . \quad (1)$$

On the prime vertical:—

$$\rho_{,} = b(1 + 2e - e\cos^2 L)$$

$$= b\left(1 + \frac{3e}{2} - \frac{e}{2}\cos 2L\right),$$ by a similar transformation to that already given.

$$\therefore \quad \text{arc of } 1' = \frac{\pi b}{180 \times 60}\left(1 + \frac{3e}{2} - \frac{e}{2}\cos 2L\right)$$

$$= \frac{\pi b}{180 \times 60 \times 584 \cdot 2}(587 \cdot 2 - \cos 2L), \text{ working as before}$$

$$= 10 \cdot 384\,(587 \cdot 2 - \cos 2L)$$

$$= 6097 \cdot 5 - 10 \cdot 4\cos 2L \text{ feet, nearly.} \quad . \quad . \quad . \quad (2)$$

We may also work in another way, thus:—

$$3\rho_{,} = b(3 + 6e - 3e\cos^2 L)$$

$$\rho = b(1 + 2e - 3e\cos^2 L)$$

$$\therefore \quad 3\rho_1 - \rho = b(2 + 4e)$$

$$\therefore \quad \rho_{,} = \frac{2b(1 + 2e) + \rho}{3}$$

$$\therefore \quad \text{arc of } 1' = \frac{\pi \rho_{,}}{180 \times 60} = \frac{\dfrac{\pi \times 2b(1+2e)}{180 \times 60} + \dfrac{\pi \rho}{180 \times 60}}{3}$$

$$= \frac{12215 \cdot 7 + \text{minute of arc on meridian}}{3}$$

This gives the value of $1'$ of arc on the prime vertical in terms of that on the meridian.

On a plane inclined θ° to meridian,

$$\rho_{,,} = \frac{\rho \rho_{,}}{\rho \sin^2 \theta + \rho_{,} \cos^2 \theta} = \frac{\rho \rho_{,}}{\dfrac{\rho + \rho_{,}}{2} + \dfrac{\rho_{,} - \rho}{2} \cos 2\theta}$$

Now

$$\rho \rho_{,} = b^2 (1 + 2e - 3e \cos^2 L)(1 + 2e - e \cos^2 L)$$

$= b^2 (1 + 4e - 4e \cos^2 L)$, neglecting terms involving e^2, which will be so small as to be negligible.

Again, we can show that

$$\left(\frac{\rho + \rho_{,}}{2}\right)^2 = b^2 (1 + 4e - 4e \cos^2 L), \text{ neglecting } e^2,$$

and

$$\left(\frac{\rho_{,} - \rho}{2}\right)^2 \cos^2 2\theta = b^2 e^2 \cos^4 L \cos^2 2\theta, \text{ which is negligible.}$$

$$\therefore \quad \left(\frac{\rho + \rho_{,}}{2}\right)^2 - \left(\frac{\rho_{,} - \rho}{2}\right)^2 \cos^2 2\theta = b^2 (1 + 4e - 4e \cos^2 L)$$

$$= \rho \rho_{,} \text{ very nearly.}$$

Substituting this value for $\rho \rho_{,,}$ we get, by factorising,

$$\rho_{,,} = \frac{\rho + \rho_{,}}{2} - \frac{\rho_{,} - \rho}{2} \cos 2\theta$$

$$\therefore \quad \text{arc of } 1' = \frac{\pi \rho_{,,}}{180 \times 60} = \frac{\pi}{180 \times 60} \left(\frac{\rho + \rho_{,}}{2} - \frac{\rho_{,} - \rho}{2} \cos 2\theta\right)$$

$=$ mean value of arcs on prime vertical and meridian
$-$ half difference of same arcs $\times \cos 2\theta$

Hence we see that in latitude L, the arc subtending one minute, or intercepted between two plumb-lines making with each other an angle of $1'$, is given by the following formulæ :—

(1) On meridian (i.e. $1'$ of latitude)

$$6077 - 31 \cos (2 L) \text{ feet}$$

(2) On the prime vertical—

$$6097 - 10 \cos 2 L \text{ feet}$$

or

$$\frac{12216 + \text{minute of latitude}}{3}$$

(3) On a great circle making an angle θ with meridian—

$$\frac{\text{minute on meridian} + \text{minute on prime vertical}}{2}$$

$$- \cos 2\,\theta \times \frac{\text{minute on prime vertical} - \text{minute on meridian}}{2}$$

For all but the most refined geodetical purposes, however, the earth may be treated as a sphere having the following dimensions :—

Length of an arc equal to radius or mean radius of the earth	=	20,890,172	feet	log 7·3199420
Length of arc subtending 1°	=	364,602	„	„ 5·5618194
Length of arc subtending 1′	=	6076·7	„	„ 3·7836681
Length of arc subtending 1″	=	101·28	„	„ 2·0055168

Great Circle, Azimuth, and Bearing.

In the chapter on Traverse Surveying, Part I., the expressions 'Great Circle,' 'Azimuth,' and 'Bearing,' have been mentioned, and to some extent defined. Before, however, proceeding to the numerical consideration of the effect of the spherical form of the earth on Surveys, it is well to recapitulate these definitions.

A 'great circle' is any circle whose plane contains the centre of a given sphere, and consequently, whose centre coincides with the centre of such sphere. It is, moreover, the largest circle that can be traced on the surface of a given sphere. The 'equator' is a 'great circle,' as also are all 'meridians.' 'Parallels of latitude' are not 'great circles.' A straight line as ranged out on the earth's surface by the surveyor, is a portion of the 'great circle.' A 'great circle' may be traced between two points on a terrestrial globe, by stretching a cord between them, and an arc of a 'great circle' is the shortest distance between any two points on the earth's surface. The 'azimuth' of a straight line is the angle which it makes with the 'great circle' of the 'meridian' passing through a point in the same.

The 'azimuth' of one station B, as determined from A, is the angle which the plane passing through A, B, and the centre of the earth, makes with the 'meridian' at A. A glance at a terrestial globe will show that the 'azimuth' of B as determined from A, will not be identical with the 'azimuth' of A as determined from B, unless A and B are either situated on the same 'parallel of latitude' or are due 'north and south' of each other.

The 'bearing' of a line is the angle which the plane of a 'great circle' containing it, makes with some standard 'great circle,' usually a meridian passing through the middle of the area to be surveyed. Thus, if A be a point on the standard meridian, the 'bearing' of another point B will be identical with the 'azimuth' of the same point as determined from A. But though the 'bearing' of A from B will be the arithmetical supplement of the bearing of B from A, the 'azimuth' of A from B, is not the supplement of that of B observed at A, but will differ by an amount depending upon the 'difference of longitude' of the points at which the reciprocal 'azimuths' are observed.

Thus, in fig. 44, if P be the pole, the azimuth of B from A is the angle between the 'meridian plane' P A O, and the plane B A O of the 'great circle' through A and B.

Similarly, the 'azimuth' of A from B is the angle between the plane P B O and the plane A B O containing A and B.

Now if the meridian planes P A O and P B O were *parallel*, the angles they would make with the intersecting plane A B O would be equal or supplementary.

But as a matter of fact they *are not* parallel, but inclined to one another at an angle equal to the difference of longitude between A and B. Hence the reciprocal azimuths are not supplementary.

For 'bearings,' on the contrary, one 'standard meridian plane' (passing preferably through the middle point of A B), is the 'plane of reference,' and the 'bearing' of A from B, or of B from A would be reckoned from planes, both parallel to this standard plane of reference, hence the 'bearings' *would be* supplementary.

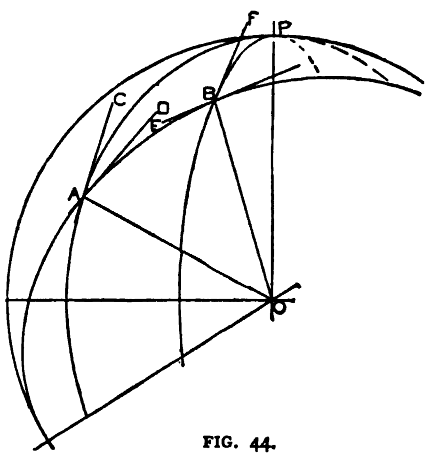


FIG. 44.

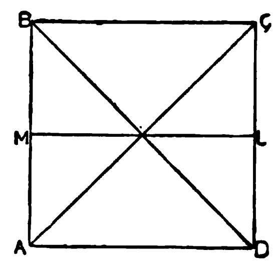


FIG. 45.

Considerations for small portions of the Earth's Surface.

In considering the geodesy of any small portion of the earth's surface (say of an area two degrees square), it is amply sufficient to treat this surface as a portion of a true sphere, having a radius of curvature equal to the *mean* between the radius of the 'meridian' and that of the 'great circle' perpendicular thereto, calculated for the 'middle latitude' of the area.

In the investigations which follow, and in the numerical examples, the mean radius of curvature for the 'middle latitude' of the place will be assumed to be 20,889,000 feet, corresponding to the *mean* 'radius of curvature,' at a place whose middle latitude is about 30°.

Numerical Examples.

Let us consider the following case. From a point A, and *in any direction whatsoever*, let a line A B, fig. 45, be ranged out 364,582 feet in length, that is, 60 nautical miles, and therefore subtending one degree, at the centre of the earth. At A and B respectively let the lines A D, B C, be exactly set out, perpendicular to A B, and prolonged

to the points D and C, both distant 364,582 feet from A B. C and D are thus fixed with regard to A B, and three portions of great circles have been set out,

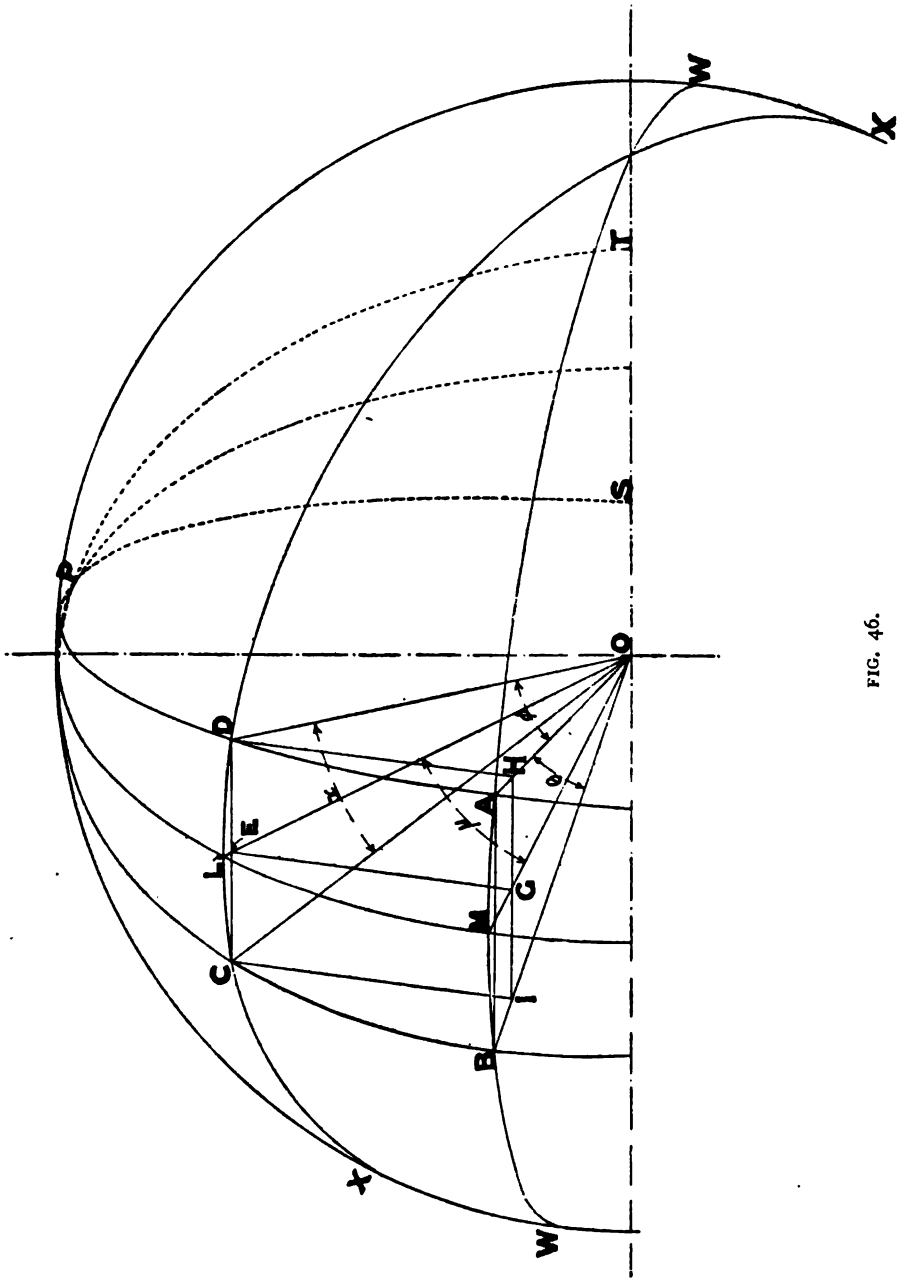


FIG. 46.

each subtending one degree, at the earth's centre, and A D and B C moreover, cutting A B at right-angles. If C D were ranged out as a straight line, joining

these points, it would be an arc of a great circle, but *not* cutting A D and B C at right-angles, and *not* equal in length to the distance A B, as would be the case if the surface of the earth were a plane.

Difference between Plane and Spherical Computation.

Now, to compute the differences between the plane and spherical assumptions, refer to fig. 46, which is an orthographic projection of a sphere, showing the arcs of great circles. The quadrilateral A B C D appears as if it were drawn on the surface of a transparent sphere, or as if the sides thereof were represented by circular hoops of thin wire. The great circle A B, is shown, purposely, with no connection to the apparent pole of the sphere. The quadrilateral drawn represents a much greater proportion of the surface of the sphere than the actual area of the case taken as an example. It is nearly 30° square, instead of 1°. This exaggeration is necessary, in order to show the proportions of the several arcs.

Now A D and B C are parts of the great circles, A D P S and B C P T, which cut each other in P, the pole of the great circle W B A W, and which is intersected by them at right-angles. C D is also a part of a great circle X C D X, oblique to W B A W.

Bisect B A in M, C D in L, and join L M. This line is perpendicular to A B, and is consequently a part of a great circle cutting the arcs B C and A D prolonged in the common pole P.

From the several points A, B, C, D, L and M, draw lines to O, the centre of the sphere. Draw the geometrical straight line joining the points C D, and therefore falling *within* the sphere, not on the surface thereof. Bisect it in E by the radius L O. From C, E and D, draw C I, E G and D H, perpendicular to the radii B O, M O and A O. These lines lie in the planes of the great circles of which B C P, M L P and A D P are quadrants, for these circles are perpendicular to the plane of the great circle of which B O A is a sector. Join the points I G H. Now the several curved lines on the earth's surface are proportional to the angles which they subtend at the centre of the sphere. Thus points I, G and H, evidently lie in one straight line, which is intersected at right-angles by the line M G O, so that I G is equal to G H. Similarly, the chord C D is bisected at right-angles at E by the line L E O. Owing to the spherical form of the earth, the geodetic straight line joining A B is represented in the figure by the arc A B, which may be measured by the angle A O B, subtended by it at the centre of the earth. Similarly, the lines B C and A D may be measured by the angles B O C and D O A. Now, C D is measured by the angle C O D, the middle distance L M by the angle L O M, and the diagonals A C and B D (not drawn in the figure) by the angles A O C and B O D respectively. It is evident that since by assumption $AM = \frac{AB}{2}$ and $CL = \frac{CD}{2}$, the $\angle BOM = \angle MOA$

$$= \angle \frac{BOA}{2}, \text{ and } \angle COL = \angle LOD = \angle \frac{COD}{2}.$$

Now put the $\angle AOB$ = angular measure of $AB = \theta$

$\angle BOC = AOD =$ „ „ $BC, AD = \phi$

$\angle$L O M = angular measure of L M = μ
$\angle$D O C = " " D C = χ

Radius of earth = ρ (at the point in question).

Then

$$\frac{OH}{OD} = \cos DOA = \cos\phi$$

$\therefore$ $OH = \rho\cos\phi$

Also

$$\frac{GH}{OH} = \sin MOA = \sin\frac{\theta}{2}$$

$\therefore$
$$GH = OH \,.\, \sin\frac{\theta}{2}$$
$$= \rho\cos\phi \,.\, \sin\frac{\theta}{2}$$

But

$$GH = ED$$

and

$$\frac{ED}{OD} = \frac{ED}{\rho} = \sin EOD = \sin\frac{\chi}{2}$$

$\therefore$ $ED = \rho\sin\frac{\chi}{2}$

But

$$ED = GH = \rho\cos\phi \,.\, \sin\frac{\theta}{2}, \text{ as above.}$$

$\therefore$
$$\sin\frac{\chi}{2} = \cos\phi \,.\, \sin\frac{\theta}{2} \quad \ldots\ldots \quad (1)$$

Again,

$$\frac{EG}{OG} = \tan MOL = \tan\mu$$

and

$$EG = DH = OD\sin DOA = \rho\sin\phi$$

Also

$$OG = OH\cos\frac{\theta}{2}$$
$$= \rho\cos\phi \,.\, \cos\frac{\theta}{2}, \text{ since } OH \quad \rho\cos\phi$$

therefore

$$\tan\mu = \frac{EG}{OG} = \frac{\sin\phi}{\cos\phi \,.\, \cos\frac{\theta}{2}} = \tan\phi \,.\, \sec\frac{\theta}{2} \quad \ldots \quad (2)$$

From these equations the length of C D and L M can be calculated. In the case under consideration, $\theta = \phi = 1^\circ$.

Hence, from equation (1)

$$\sin\frac{\chi}{2} = \cos 1^\circ \sin 30'$$

whence we obtain

$$x = 59' \ 59{\cdot}452''$$
$$= 3599{\cdot}452''$$

Multiply this by the number of feet in one second, i. e. by

$$\frac{\pi \times 20{,}889{,}000}{180 \times 60 \times 60}$$

and the length of the arc C D becomes	364,526 feet
Whereas if the earth's surface were flat, it would be equal to A B, or	364,582 „
Hence the value of C D, if calculated by plane geometry, would be too long by the difference, or . .	56 feet

Thus, by neglecting the spherical shape of the earth, an error is made in the length of C D of about 56 feet, or an average rate of nearly ·93 foot per nautical mile.

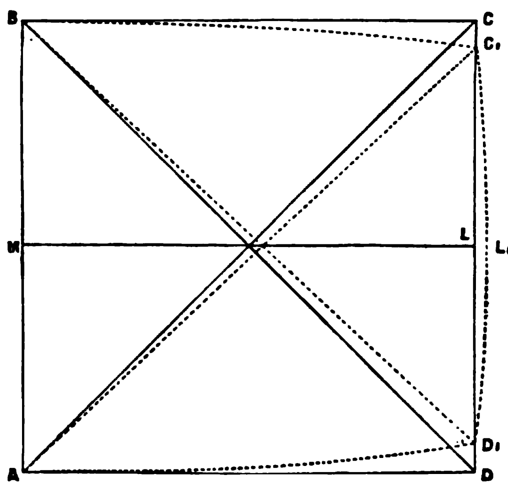


FIG. 47.

Again, from equation (2)

$$\tan \mu = \tan 1^\circ \sec 30'$$

whence

$$\mu = 1^\circ \ 0' \ 0{\cdot}138''$$
$$= 3600{\cdot}138''$$

Reducing to feet as before,

Arc L M = . . .	364,596 feet
By plane geometry	364,582 „
Showing a deficit of	14 feet

Hence, the arc L M calculated by plane geometry would be 14 feet *too short* being an average rate of about ·23 of a foot per mile.

The 'plane' and 'spherical' values of the several angles representing the 'diagonals' may be similarly compared by the ordinary rules of plane and spherical trigonometry. The values obtained in degrees, minutes, and seconds, may then be reduced to feet (by multiplying the angle in seconds by the number of feet subtending an angle of one second at the earth's centre), and compared with the values obtained by plane trigonometry.

The following table gives the results of the computation showing excess or defect due to calculation by plane trigonometry, and refers to fig. 46.

EXCESS OR DEFECT BY PLANE TRIGONOMETRY.

—	Plane.	Spherical.	—
A B = B C = A D	364,582	364,582	..
Side C D	364,582	364,526	+ 56 feet.
Diagonals A C, B D	515,596	515,583	+ 13 „
Middle distance L M	364,582	364,596	− 14 „
Angles A B C, B A D	90° 0′ 0″	90° 0′ 0″	..
„ A B D, B D A, A C B, B D C .	45° 0′ 0″	45° 0′ 16″	− 16 secs.
„ C A D, D C A	45° 0′ 0″	44° 59′ 44″	+ 16 „
Sum of three angles of any one of triangles A B D, etc.	180° 0′ 0″	180° 0′ 32″	− 32 „
Sum of angles of rectangle A B C D .	360° 0′ 0″	360° 1′ 4″	− 64 „

The difference between the plane and spherical lines is shown in a highly exaggerated manner by the full lines and dotted lines in fig. 47. The spherical lines or arcs BC_1 and AD_1 are by assumption equal to A B, and are also at right angles thereto, but $C_1 D_1$ is shorter than C D by 56 feet. The middle distance $M L_1$ is 14 feet longer than M L, whereas by plane geometry it would be evidently equal to A D or B C. The spherical diagonals $A C_1$, $B D_1$, are shorter than the plane diagonals A C, B D, by 13 feet. The angles $A D_1 C_1$ and $B C_1 D_1$ are 32 seconds in excess of A D C and B C D respectively, which by plane geometry would be right angles. It will, however, be seen that the differences are trivial, even in so large an area as 60 by 60 square (nautical) miles, and are smaller than the probable error of measurement. They would also be inappreciable on paper, if the area were plotted to any reasonable scale. Even if a scale large enough to make these differences appreciable were adopted, the several lines, as calculated by spherical trigonometry, could not be laid down so as to agree with each other in every respect.

Spherical Excess of Triangle.

In the above example, it will be observed that the sum of three angles of any of the spherical triangles into which the figure A B C D is divided exceeds two right angles by a certain number of seconds.

This is a universal property of all spherical triangles, and the excess above 180° is called the 'spherical excess' of the triangle.

It is found by the following rule:—

'*The sum of the three angles of any spherical triangle exceeds two right angles by an angle which bears the same ratio to four right angles as the area of the triangle bears to the area of a hemisphere.*'

Let A = area of triangle in square feet.
r = radius of sphere in feet.
E = spherical excess in seconds of arc.

Then area of a hemisphere $= 2\pi r^2$

and $E : 360° :: A : 2\pi r^2$

$$\therefore \quad E = \frac{A \times 180°}{\pi r^2} \quad . \quad . \quad . \quad . \quad (1)$$

In using this formula, it will be amply accurate for present purposes to compute the area A by plane trigonometry.

A simple formula may be deduced to compute the spherical excess of any triangle from its area in square miles, acres, feet, or any other units.

The radius of the perfect sphere which has been taken as most nearly approximating the form of the earth, is about 20,889,000 feet.

Sum $= \log r^2$	Log r in feet	$= 7 \cdot 319\ 9176$
	Log r	$= 7 \cdot 319\ 9176$
Log π	. . . .	$= 0 \cdot 497\ 1499$
	Log πr^2	$= 15 \cdot 136\ 9851$
Colog πr^2 (deducting from zero)	. .	$= \overline{16} \cdot 863\ 0149$
Log 180° in seconds	. . . . .	$= 5 \cdot 811\ 5750$
$\therefore$ Constant log or log $\frac{E}{A}$ (for sq. ft.) is	. .	$\overline{10} \cdot 674\ 5899$

For example, the area of the triangle A B C, fig. 46 is

$$\frac{364582 \text{ ft.} \times 364582}{2} \text{ sq. feet.}$$

Colog $2 = \log 0 \cdot 5$	$= \overline{1} \cdot 698\ 9700$
Log 364,582	$= 5 \cdot 561\ 7950$
Ditto for square	$= 5 \cdot 561\ 7950$
Constant log as above	$= \overline{10} \cdot 674\ 5899$
$\therefore$ Spherical excess of this triangle $= 31 \cdot 4$ seconds	$1 \cdot 497\ 1499$

It is perhaps more convenient to have the constant for areas in square miles.

Log constant for feet	$\overline{10}\cdot674\ 5899$
Log 5280	$3\cdot722\ 6339$
Ditto for square	$3\cdot722\ 6339$
Constant log or log E for 1 square mile .	$\overline{2}\cdot119\ 8577$

Antilog = $0''\cdot0132$

From the above it will be seen that the spherical excess of a triangle having an area of one square mile is less than one seventieth of a *second*—a *negligible* quantity in ordinary surveying operations.

Any polygon may be divided into triangles by lines radiating, from a point or points within it, to its angles. The three angles of each triangle will have the spherical excess proper to its area. The angles between the lines which cut each other in the central point will sum to 360°. The sum of the interior angles of the polygon, added to four right angles, will exceed twice as many right

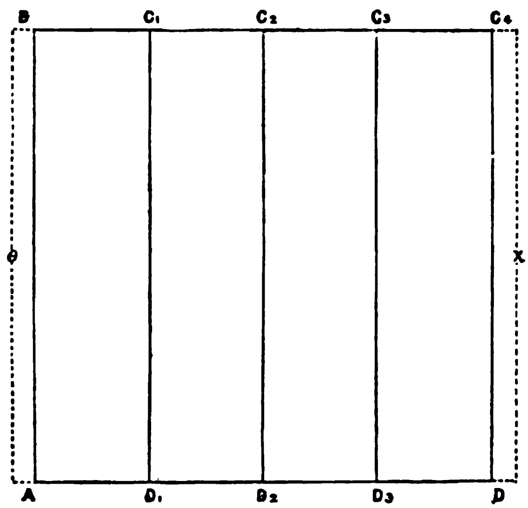


FIG. 48.

angles as the figure has sides, by the sum of the spherical excess of the triangles into which the polygon has been divided, that is to say, the spherical excess of the polygon will be that due to its area as if calculated for a single triangle.

The spherical excess of the figure which has been dealt with, viz. 60 square (nautical) miles in area, amounts to 62 seconds. If, instead of a square, a polygon of equal area had been surveyed by traversing, there would have been a closing error in the angles amounting to $1'\cdot2$, even supposing absolute accuracy in the work. As the perimeter would be about 240 miles, the probable number of sides would not be less than 240. In this case the spherical excess would amount to about one quarter of a second per side.

This is vastly less than the probable error of an angular observation of any ordinary theodolite.

Before quitting this subject, it will be well to ascertain the law which governs

the error due to the convergence of apparently parallel straight lines, at different distances from the straight line to which they are mutually perpendicular.

Law of Convergence. Let B C and A D (fig. 48) be perpendicular to A B as before. It has been shown that if A B, B C and A D are each 60 nautical miles in length, C D is 56 feet shorter than A B, or at the rate of about ·93 of a foot per mile. It is desirable to ascertain what will be the result, for lines $C_1 D_1$, $C_2 D_2$, etc., at varying distances from A B, and for any given value of A B.

Putting $AB = \theta$ in angular measurement

$BC = AD = \phi$,

and $CD = \chi$, as before,

then, as we have seen,

$$\sin \frac{\chi}{2} = \sin \frac{\theta}{2} \cdot \cos \phi. \quad . \quad . \quad . \quad . \quad (2)$$

Now the sines of angles less than about one degree, differ very little from their circular measure, or $\frac{\text{arc}}{\text{radius}}$.

Hence for such small angles we may safely put $\frac{\text{length of arc in feet}}{\text{radius of earth in feet}}$ for the sine of the angle subtended by the arc.

For example,

The nat. sine of 1°	=	·0174524
$\frac{\text{arc}}{\text{radius}}$ for 1°	=	·0174533
Difference		·0000009

or only 8 parts in 10 millions.

Now let ρ = radius of earth in feet at point in question. Then we may write

$$\sin \frac{\theta}{2} = \frac{AB}{2} \div \rho = \frac{AB}{2\rho}$$

and

$$\sin \frac{\chi}{2} = \frac{CD}{2} \div \rho = \frac{CD}{2\rho}$$

∴ the above equation (2) becomes

$$\frac{CD}{2\rho} = \frac{AB}{2\rho} \cos \phi$$

or

$$\frac{CD}{AB} = \cos \phi \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, we see that the ratio of C D to A B is independent of the length of A B, provided the latter be small enough for us to take $\frac{\theta}{2} = \sin \frac{\theta}{2}$, and that it varies solely as the cosines of the angles subtended by A D and B C at the centre

of the sphere. Thus, when A B, B C, and A D, are each 60 nautical miles long, C D is shorter than A B by 56 feet, or at the rate of (as before stated) ·93 of a foot per mile.

If, however, the distance A B were one, two, or three nautical miles, B C and A D remaining 60 nautical miles, the convergence (A B − D C) would be as follows:—

$$\begin{aligned} &AB,\ 1\text{ mile,}\quad AB - D_1C_1 = 1 \times \cdot 93 = \ \cdot 93 \text{ foot} \\ &AB,\ 2\text{ miles,}\ AB - D_2C_2 = 2 \times \cdot 93 = 1\cdot 86\ \text{,,} \\ &AB,\ 3\ \text{,,}\quad AB - D_3C_3 = 3 \times \cdot 93 = 2\cdot 79 \text{ feet} \end{aligned}$$

and so on.

Now from equation (3) above

$$\frac{AB - CD}{AB} = \frac{1 - \cos\phi}{1} = 1 - \cos\phi$$

But $\frac{AB - CD}{AB}$ is the *rate* of convergence, expressed as a decimal fraction. or per unit length of A B.

Now, multiplying by the number of feet in a nautical mile, the *rate* of convergence in feet per nautical mile is obtained.

Thus, *rate* of convergence in feet per nautical mile = $(1 - \cos\phi) \times 6076$.

The following table gives the results of this calculation:—

TABLE A.—Showing Convergence of Parallel 'Straight Lines' One Nautical Mile Apart, on the Earth's Surface.

Value of ϕ in Seconds = angular Measure of Lines.	Natural Cosines.	1 − Cosine.	Convergence in Feet = (1 − nat. cos.) × 6076.
5′	0·9999989	·0000011	·00668
10′	0·9999958	·0000042	·02552
15′	0·9999905	·0000095	·05772
20′	0·9999831	·0000169	·10268
25′	0·9999736	·0000264	·16041
30′	0·9999619	·0000381	·23150
35′	0·9999482	·0000518	·31474
40′	0·9999323	·0000677	·41135
45′	0·9999143	·0000857	·52071
50′	0·9998942	·0001058	·64284
55′	0·9998720	·0001280	·77773
60′	0·9998477	·0001523	·92537

An inspection of this table shows that the convergence is roughly proportional to the squares of the length of the lines. This property of convergence is one reason why the 'prime meridian' should run through the 'centre of the survey.'

If the rectangle A B C D, fig. 49, were referred to a meridian P O Q, running through the centre thereof, the rate of error in C S, S D, A B, and B R would be one-fourth only of the error in C D when referred to the meridian A B. In the case assumed, the sides being 60 nautical miles long, the rate of error will be reduced from ·93 of a foot per mile, to ·23 of a foot per mile. (From table ·23 foot = convergence when angle = 30″.)

Example in Difference between computation of Triangles by Plane and Spherical Trigonometry.

To ascertain the error, due to the computation of triangles by plane instead of by spherical trigonometry, the solution of a simple case will give the clearest conception of the difference between the two methods.

Let A C (fig. 50) be a meridian, let A B be a measured line, making a known angle C A B = θ, with the meridian. Draw C B perpendicular to A C.

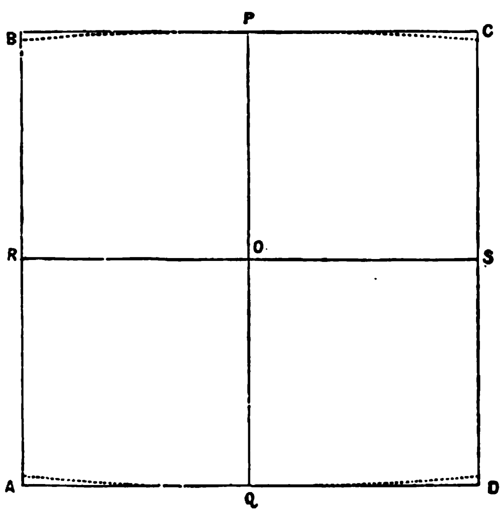


FIG. 49.

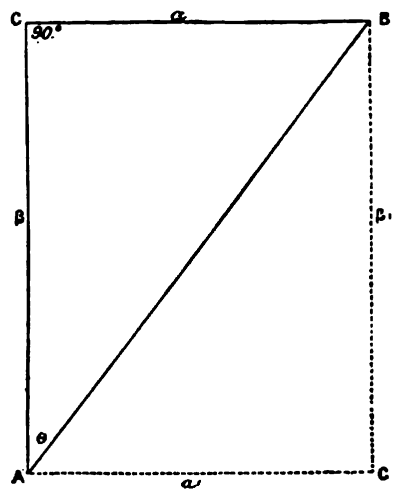


FIG. 50.

Then, by plane trigonometry,

$$A\,C = A\,B \cos\theta = \text{difference of latitude, A to B}$$
$$C\,B = A\,B \sin\theta = \text{departure, A to B}$$

Now if

$$A\,B = 1^\circ, \text{ or } 364{,}582 \text{ feet to an assumed radius,}$$

and

$$\theta = 40^\circ, \text{ say,}$$

then

$$A\,C = 364{,}582 \cos 40^\circ = 279{,}286 \text{ feet,}$$

and

$$C\,B = 364{,}582 \sin 40^\circ = 234{,}349 \text{ feet.}$$

To solve the same problem by spherical trigonometry, the distance A B must be replaced by the angle which it subtends at the centre of the terrestrial sphere,

by dividing its length in feet, by the length of an arc subtending an angle of 1°, 1′, or 1″, at the earth's centre. A C and C B will then be obtained in terms of the angles which they subtend, which can again be brought into lengths by multiplying by the length of the arc subtending the unit angle.

In the present example, the distance A B (364,582 feet) subtends an angle of 1° at the earth's centre.

Adopting the usual notation of spherical trigonometry, let the

$$\begin{aligned} \text{Angle subtended by } CB &= \alpha \\ \text{,, \quad ,, \quad } AC &= \beta \\ \text{,, \quad ,, \quad } AB &= \gamma \\ \text{Angle } CAB &= \theta \end{aligned}$$

Then $\left.\begin{aligned} \tan\beta &= \tan\gamma \,.\, \cos\theta \\ \sin\alpha &= \sin\gamma \,.\, \sin\theta \end{aligned}\right\}$ by spherical trigonometry (p. 83)

$\therefore$ since $\gamma = 1°$

$\tan\beta = \tan 1° \cos 40°$

$\therefore$ $\beta = 0° \, 45' \, 57{\cdot}89''$, which, reduced to feet, gives

A C = 279,299 feet, by spherical trigonometry
as against 279,286 „ by plane trigonometry

showing a deficit of 13 „ by plane as against spherical trigonometry

Again,

$$\begin{aligned} \sin\alpha &= \sin\gamma \,.\, \sin\theta \\ &= \sin 1° \times \sin 40° \end{aligned}$$

$$\therefore \quad \begin{aligned} \alpha &= 38' \, 34{\cdot}16'' \\ &= 2314{\cdot}16'' \end{aligned}$$

Reducing to feet as before,

B C = Departure = 234,361 feet by spherical trigonometry,
as against 234,349 „ by plane trigonometry.

Showing a deficit of 12 „ by plane, as against spherical trigonometry.

If we complete the parallelogram A C B C_1, then by plane geometry A C = B C_1, and C B = A C_1.

Now, if we consider that B C_1 and A C_1 are the spherical 'differences of latitude' and 'departures' respectively, then the arcs A C and B C_1 are parts of great circles and cut the great circle of which A C_1 is a segment at right-angles. We have, therefore, a right-angled spherical triangle A B C_1 in which the angle at A is $90° - \theta$.

Calculating in this manner, we get somewhat different values for 'differences of latitude' and 'departure' from those calculated from the plane triangle A B C. Thus, in the example, B C_1 = 'difference of latitude' = 279,283 feet, by spherical trigonometry, as against 279,286 feet by plane trigonometry, showing a difference of 6 feet, A C_1 = 'departure' = 234,382 feet by spherical trigonometry, as against 234,349 feet by plane trigonometry, showing a difference of 14 feet. For shorter distances the differences will be less in proportion.

Up to this point it is evident that the error both in computation and plotting, due to the assumption that the area surveyed is a 'plane,' and not a 'segment of a sphere,' is wholly negligible in comparison with the unavoidable errors of any ordinary method of surveying, conducted with the instruments which the surveyor usually possesses, and amounts to a quantity which could not be detected when the survey is plotted to the largest scale practicable.

Convergence of Meridians discussed.

Now, the last point which has to be examined is whether if B C, fig. 51, be a 'north-and-south' line, and A D be by plane geometry a line parallel to it, A D may also be taken to be a 'north-and-south' line. It will be found that this assumption is not permissible, even in surveys of moderate extent.

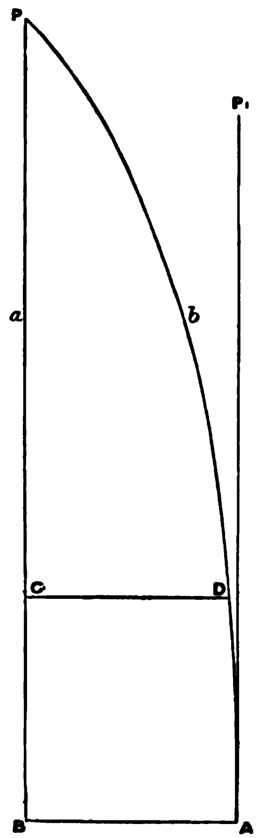


FIG. 51.

B C is a portion of a terrestrial 'meridian' passing through the earth's pole. Now all *true* 'north-and-south' lines are portions of 'meridians,' and, therefore, converge and cut each other in the earth's poles. B C and A D are by assumption, at right-angles to A B. They are 'great circles,' and as has been shown, converge, not to the terrestrial pole, but to the pole of the 'great circle' of which A B is a segment. If A B were a part of the 'equator,' then B C and A D *would* converge to the terrestrial pole 90° distant. If B were in any higher latitude, the meridian through A would cut that through B in the terrestial sphere's pole, *not* in the pole of A B, but at a smaller distance from B than 90°. The distance B P would be 90° minus the latitude of B, and if A B be perpendicular to B P, the angle between A B and the meridian of A will be less or greater than 90°. We have, therefore, a right-angled spherical triangle, in which the right angle is at B, and the sides A B and B P are given, whilst the angle at A is sought.

Let the angle subtended by $AB = p$

" " $BP = a$

" " $AP = b$

Then, by spherical trigonometry,

$$\tan A = \frac{\tan a}{\sin p} \text{ (p. 83)}$$

But

$$a = 90^\circ - \text{latitude of B}$$

$$\therefore \quad \tan A = \frac{\text{cot latitude of B}}{\sin p}$$

and

$$P_1 A D = 90^\circ - B A P = 90^\circ - A$$
$$= \text{convergence of meridians}$$

$$\therefore \quad \text{cot convergence} = \tan A = \text{cot latitude} \times \text{cosec}\, p \quad . \qquad (4)$$

If $\quad A B = 60$ nautical miles, so that $p = 1^\circ$

and if $\quad$ latitude of $B = 45^\circ$, we have

$$\textit{cot convergence} = \textit{cot}\ 45^\circ\ \textit{cosec}\ 1^\circ$$

whence

$$\textit{convergence} = 59'\ 59''\cdot 45$$

Now it must be observed that this angle is far greater than the probable error in the determination of the *true* north with a common theodolite. If B were in a lower latitude, say 15°, we should have less convergence, but still an appreciable amount, namely, $16'\ 4''$.

Now, when the departure is not great, but under 60 miles, this expression may be simplified, so that the convergence of the meridians may be calculated, without the reduction of length to angle.

For, since $\tan = \frac{1}{\cot}$ and $\sin = \frac{1}{\text{cosec}}$, formula (4) above may be written

$$\text{tan convergence} = \text{tan latitude} \times \sin p$$

but for small angles, the sines, tangents, and circular measure, i.e. $\frac{\text{arc}}{\text{radius}}$, may be taken as equal.

Hence

$$\sin p = \frac{\text{departure}}{\text{radius of earth}}, \text{ if } p \text{ be small}$$

and

$$\text{tan convergence} = \text{circular measure of convergence}$$
$$= \text{convergence in minutes} \times \text{circular measure of } 1'$$

$$\therefore \quad \text{convergence in minutes}$$
$$= \frac{\text{tan convergence}}{\tan 1'}$$
$$= \frac{\text{tan latitude} \times \sin p}{\tan 1'}$$
$$= \frac{\text{tan latitude} \times \text{departure}}{\text{radius of earth} \times \tan 1'}$$

The 'departure' and the 'radius' must be in the same units.

The logarithmic computation, by this formula, would be as follows:

To a constant logarithm . . . . .
add log tan lat . . . . .
and log departure . . . . . ______
then log (convergence in minutes) = ______

The constant log is that of

$$\frac{\text{unit in which 'departure' is expressed reduced to feet}}{\tan 1 \times \text{radius of earth in feet}}$$

Thus, if 'departure' be in nautical miles,

$$1 \text{ nautical mile} = \frac{\pi \rho}{180 \times 60} \text{ feet, where } \rho \text{ is in feet,}$$

$$\text{and } \tan 1 = \frac{\pi}{180 \times 60}$$

$\therefore$ the above fraction $= 1$, and constant log $= 0$

Thus, for departure in nautical miles,			constant log	$= 0$
"	"	statute "	"	$= \bar{1}\cdot 9390$
"	"	chains of 66 feet	"	$= \bar{2}\cdot 0359$
"	"	feet	"	$= \bar{4}\cdot 2164$

Example:—

Departure = 60 nautical miles, latitude 45°:

$$\begin{aligned} \text{Constant log} &= 0\cdot 0000 \\ \log \tan 45^\circ &= 0\cdot 0000 \\ \log 60 &= 1\cdot 7782 \\ \hline & 1\cdot 7782 = \log 60 \end{aligned}$$

$\therefore$ convergence = 60′

With a 'departure' of 12 statute miles in latitude 50°:

$$\begin{aligned} \text{Constant log} &= \bar{1}\cdot 9290 \\ \log \tan 50^\circ &= \cdot 0762 \\ \log 12 &= 1\cdot 0792 \\ \hline & 1\cdot 0944 = \log 12\cdot 43 \end{aligned}$$

$\therefore$ convergence $= 12'\cdot 43$

Now, this convergence of meridians has no connection with any distortion, caused by neglecting the spherical form of the earth. It is an *actual fact*, and would be observed if the theodolite were set up and the true 'north-and-south' line were determined. Thus, referring to fig. 51, assume as before that B C is a part of a meridian, and the angle A B C a right angle, then, if the *true* north were determined by a theodolite set up at D, the true 'north line' found in 'latitude' 45° north would not be parallel to B C, or at right angles to A B, but would make an angle of 89° with it, whereas the distortion in the angle due to neglecting the spherical form of the earth is but 32″ in any latitude: that is to say that by plane geometry, the angle at D is made 90°, instead of 90° 0′ 32″ as it would have been by spherical geometry.

The 'convergence of the meridian' can be observed in any map of an extended area of the globe's surface, and even in the relatively small area which is represented by a sheet of the Ordnance Survey of England, on the scale of 6 inches to one mile. In these maps the degrees, minutes, and seconds of longi-

tude, are figured both on the upper and lower margins. It will be noticed that there are a greater number of degrees, minutes, and seconds, marked along the upper margin of the paper than along the bottom. The true 'north-and-south' line through any point is, therefore, not a line parallel to the lateral margins but one drawn through the point, and intercepting equal 'longitudes' at the top and bottom of the sheet.

The above considerations show that *the errors due to the neglect of the 'spherical form of the earth,'* as regards the computation of distances, and the projection of points, are *negligible* as compared with the inevitable errors of ordinary means of measurement in the field, and of projection on paper. Even if the survey were made with the most accurate methods known to science, computing 'latitudes' and 'departures' by spherical geometry, the points so obtained could not be projected upon the paper without distortion in one direction or the other. The only plan available is to reduce distortion to a minimum, by dividing an extensive survey into districts of moderate size, and to refer the 'latitudes' and 'departures' of the several points to a *true* 'north-and-south' line running through the centre of each district. The relative positions of the points on each sheet would suffer but little distortion, but the several sheets will not fit together with mathematical accuracy. It is indeed impossible to arrange matters so that they will fit together in every direction without producing distortion.

On the other hand, *the errors due to the neglect of the 'convergence of meridians,'* must not be disregarded, for this is an actual fact, both on the surface of the earth, and on the plan or map.

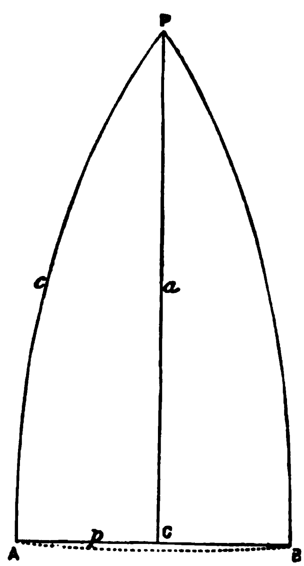


FIG. 52.

Examples on Convergence of Meridians, Bearing, and Azimuth.

The case of setting out a parallel of latitude as a boundary betwen two states, affords an instructive example of the difference between 'bearing' and 'azimuth' and of the 'convergence of the meridians.'

Suppose that it be desired to set out the parallel of latitude 54° north, a place in exactly that 'latitude,' being given as a starting-point. It is desired to range out a straight line from the starting-point, in such direction that the extremity thereof shall also be in latitude 54°. Required, the 'azimuth' of the line at the starting-point, its intended length being 364,580 feet (60 nautical miles nearly). Now this distance subtends 1° at the centre of the earth. In fig. 52 let P be the pole, A and B the terminal stations. We have the spherical triangle P A B in which the arcs P A and P B = 90°, − latitude, or co-latitude

$$= 36^\circ \; 0' \; 0''$$

$$\text{and arc } AB = \;\; 1^\circ \; 0' \; 0''$$

From these data the angles at A and B may be calculated by the rules for right-angled spherical triangles, as follows. Bisect the arc A B in C. Draw a great circle through P C cutting A B in C. Then P A C is a right-angled triangle, in which C is the right-angle, and in which the hypothenuse c and the side p are given, and the angle A is sought.

The formula is

$$\cos A = \cot c \tan p \qquad \text{(p. 83)}$$

		Log.
$c = 36^\circ\ 0'\ 0''$,	$\log \cot 36^\circ\ 0'\ 0''$ . . =	$10 \cdot 138\ 7390$
$p = 0^\circ\ 30'\ 0''$,	$\log \tan 0^\circ\ 30'\ 0''$ . . =	$7 \cdot 940\ 8584$
$\therefore$ $\log \cos A$ =	$\log \cos 89^\circ\ 18'\ 42''$ . . =	$8 \cdot 079\ 5974$

and $A = 89^\circ\ 18'\ 42''$

Hence the surveyor would range out a straight line A C B, commencing with an 'azimuth' $89^\circ\ 18'\ 42''$ north-east and prolonging the same on the ground, until he had chained 364,580 feet, when astronomical observation would show that he was again in latitude 54°, and that the 'azimuth' of the line was $89^\circ\ 18'\ 42''$ north-west. This line is, of course, really a portion of a great circle, but appears straight on the ground, and is shown straight on the figure.

If he paused exactly half-way and checked his latitude, he would find that he was not in 'latitude' 54° but in a higher 'latitude.' The 'latitude' of the point C, the middle point, may be calculated from the same right-angled triangle, the side a being the co-latitude of the point C.

The formula is

$$\cos a = \frac{\cos c}{\cos p} \qquad \text{(p. 83)}$$

$\log \cos c$ =	$\log \cos 36^\circ\ 0'\ 0''$ . . =	$9 \cdot 907\ 9576$
$\log \cos p$ =	$\log \cos 0^\circ\ 30'\ 0''$ . . =	$9 \cdot 999\ 9835$
$\therefore$ $\log \cos a$ =	$\log \cos 35^\circ\ 59'\ 49''$ . . =	$9 \cdot 907\ 9741$

and $a = 35^\circ\ 59'\ 49''$

$\therefore$ latitude of point $C = 54^\circ\ 0'\ 11''$

Therefore, the point C would be $11''$ north of the true parallel of 'latitude,' or with the mean radius of the earth 1114 feet nearly. This offset being set off due south at C, would fix the middle point of the arc of the desired parallel of latitude, which is represented by the dotted line in the figure. Similarly, the offsets of other points may be calculated and set off, so as to fix the curved line, representing the 'parallel of latitude.'

The same curve might also be traced by continually setting out *very short* lengths of line, always with an azimuth of 90° east. The total distance chained would be found to be greater than in the first instance.

It is instructive to calculate the 'difference in length,' between the 'great circle' and the 'parallel of latitude' between A and B.

The first step is to calculate the 'difference in longitude' between A and B. This is the angle at the pole of the earth, A P B, which again is twice the angle A P C in the right-angled spherical triangle A P C.

$$\cot P = \cot p \sin a$$

$\log \cot p = \log \cot$	$0^\circ\ 30'\ 0''$	. .	$= 12 \cdot 059\ 1416$
$\log \sin a = \log \sin$	$35^\circ\ 59'\ 49''$	. .	$= 9 \cdot 769\ 1868$
$\log \cot P = \log \cot$	$0^\circ\ 51'\ 2 \cdot 5''$	. .	$= 11 \cdot 828\ 3284$
Multiply by 2 .	2		

$$6125'' = 1^\circ\ 42'\ 5'' = \text{diff. of longitude between A and B.}$$

The length of the arc of a small circle being proportional to the radius thereof, or to the cosine of the 'latitude,' it is merely necessary to multiply together, the number of feet in a second of arc of a great circle, the number of seconds of 'longitude,' and the cosine of the 'latitude.'

Log feet in one second	$= 2 \cdot 005\ 4925$
Log 6125 seconds.	$= 3 \cdot 787\ 1061$
Log cos $54^\circ\ 0'\ 0''$	$= 9 \cdot 769\ 2187$
∴ Log length of arc of small circle or parallel of latitude = log 364,600 feet .	$= 5 \cdot 561\ 8173$

Now, 364,582 feet is the length of arc of great circle,

∴ difference = 18 feet.

If, on the other hand, the surveyor proceeded to lay out a line from A, making each short length run exactly east (by means of a perfectly corrected compass, for example) he would reach B, and lay out a parallel of latitude correctly. But, if, confounding 'azimuth' with 'bearing,' he proceeded to plot the line as straight, he would delineate a curve as a straight line, and place the middle point about 1114 feet to the south of its proper position. He would obtain also a slightly erroneous distance between A and B.

If, on the other hand, the great circle from A to B were plotted at right-angles to the 'north-and-south' line, through the middle point C (as in the figure), the line on the map would be straight, as it would appear on the earth, whilst the points on the 'parallel of latitude' would appear to be on a curve, as they would actually be on the ground.

To show the 'azimuth' of the line, at any point on its length, it would only be necessary to draw the 'meridian' with the 'convergence' proper to the point. Thus the 'meridians' at the extremities, would make angles of $89^\circ\ 18'\ 42''$ with the line, or in other words, the 'convergence' of the 'extreme meridians' would each be $41'\ 18''$ with the central 'prime meridian.' The several points at the same 'latitude' would be obtained correctly, from a curved line, cutting each successive meridian at right-angles. The 'parallel' of 54° would be none other than the curve set out by offsets from the great circle, as already described, and

measured from this, the 'latitude' of the middle or any other point would be 54.

Suppose, again, the case of a survey for a railway in latitude of 60° north, such that the 'departure' between its extremities is 20 miles.

The surveyor begins by determining, astronomically, the 'azimuth' of the first line, from the starting point, and takes that as its 'bearing.' He measures the various angles along the line, and from them calculates the 'bearing' of the last line.

He wishes to check his angular measurements, and at the finishing point, determines the 'azimuth' of the last line, and compares it with the computed 'bearing.'

If his angular measurements, and his astronomical observations were absolutely perfect, he would, nevertheless, find a difference between the computed '*bearing*' of the last line, and its '*azimuth*,' as observed astronomically. This would be due to the convergence of the meridians, which may be thus calculated by the approximate formula—

Constant log	=	$\bar{1}\cdot9390$	(p. 112)
Log tan 60°	=	$10\cdot2386$	
Log 20	=	$1\cdot3010$	
∴ log convergence = log 30·10 .	=	$1\cdot4786$	

and convergence = 30′·10

Now, if he were to neglect this 'convergence,' and adjust his angles, so as to make the final 'bearing' of the last line agree with the observed '*azimuth*,' at the final station, he would distort his line by half a degree, or if there were twenty lines, the angles would be altered by about 1½ minutes each, a quantity quite within the limits of accuracy of an ordinary theodolite.

Again, if he were to start to survey a further section, using an 'azimuth' observed at the end of the first section, for the 'bearing' of the first line of the second section, then the relative directions of the two sections would be distorted by half a degree.

Therefore, from these examples it will be seen, that as above stated, the 'convergence of the meridians' must not be neglected, in checking angular work by astronomical observations.

It is, moreover, seen that even in so short a distance as 1 mile, in latitude 60°, the convergence amounts to 1½ minutes, a quantity in excess of the probable error of an ordinary 'azimuth' observation.

CHAPTER VIII.

PRELIMINARY INVESTIGATIONS AND PROBLEMS IN PHYSICAL AND PRACTICAL ASTRONOMY.

Object of Chapter. THE following Chapter may be regarded as introductory to Chapter IX. on Practical Astronomy, inasmuch as the data and problems in connection with the movements of the heavenly bodies as seen in the firmament are in it considered, to the extent necessary to enable the student to grasp the calculations for the determination of latitude, time, azimuth, and longitude.

It treats in fact of the *physical* problems which must be mastered, before entering on the study of *practical* astronomy.

The Earth. It is well known that the earth revolves about its axis, producing the effect of night and day, and also that the earth and planets revolve in orbits round the sun. Again, the stars, the sun, and the planets, appear to rise in the east, to attain a certain altitude, and then to set in the west.

The Fixed Stars. The stars which appear to move as though they were attached to the surface of a vast 'crystal sphere,' surrounding the earth, but at a very great distance from it, preserving their relative positions, night after night, and year after year, are known as the 'fixed stars.'

The Sun, Moon, and Planets. There are other heavenly bodies, which appear to move about amongst these 'fixed stars' in an irregular manner. These are the sun, the moon, and the planets.

Apparent Motions only, of Stars, etc. need be considered. In problems on astronomy it is only the *apparent* motions of the heavenly bodies, as above indicated, which have to be considered, as if seen by an observer on the surface of the earth.

Apparent Motions of the 'Fixed Stars.' Let us first consider the *apparent* motions of the 'fixed stars,' since these are *relatively* simple. As already stated, they appear to move, as though attached to a 'crystal sphere,' revolving about the earth. The axis of revolution of this 'stellar sphere' is very nearly identical with that of the earth produced, though as a fact the axis of revolution of the 'stellar sphere' rolls or sways a little from the earth's axis, producing slight apparent motions in the fixed stars. These are but small, however, and may be neglected for the present, being at any rate negligible for short periods of time.

Parallax of Stars and Planets.

Parallax may be defined as, 'the difference in the apparent position of a heavenly body, when viewed from the centre of the earth and from its surface,' which with the 'fixed stars' is an absolutely negligible quantity. The nearest fixed star is so distant that the mean radius of the earth's annual orbit round the sun (92 millions of miles), subtends but an angle of about one-third of a second of arc, so that the earth's radius, of about 4000 miles, subtends at the said star but a 15-millionth of a second of arc. Consequently, the position of any 'fixed star' as observed from

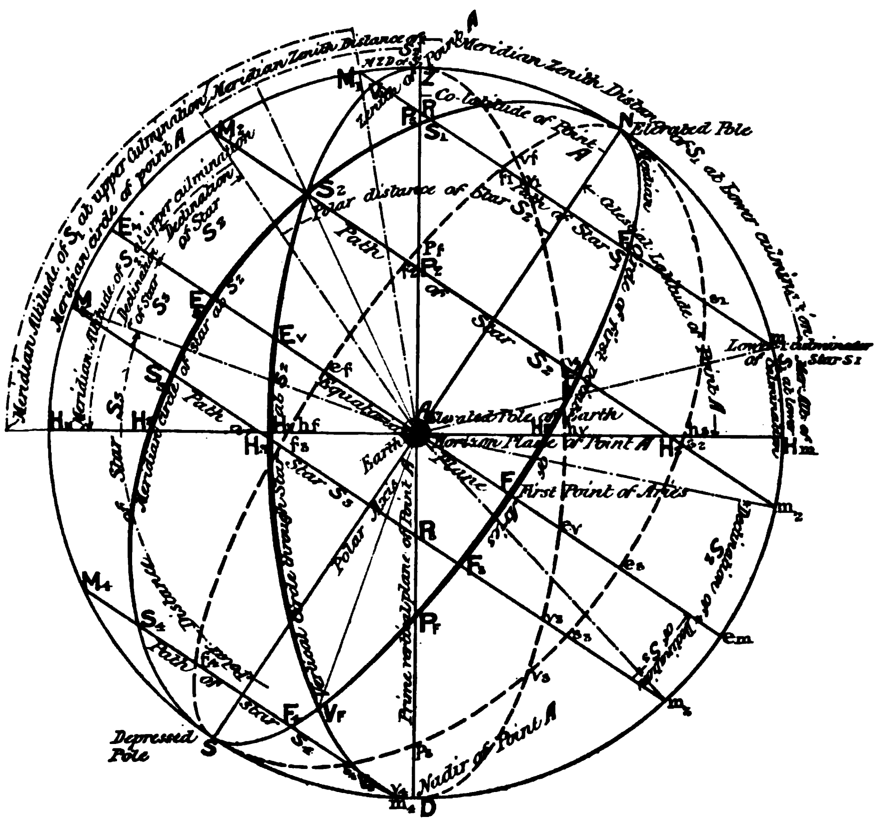


FIG. A.—ELEVATION ON MERIDIAN OF POINT A.

the earth's surface, is absolutely the same as though it were observed from the earth's centre. At the sun, however, the earth's radius subtends 8·8 seconds of arc, so that a slight correction will reduce observations of the sun made on the surface of the earth, to its centre. The moon, on the other hand, is so near to the earth that the radius of the latter subtends at the centre of the moon, nearly one degree. The co-efficient for reducing observations of the sun, moon, or planets (made at the surface of the earth) to its centre, is called the 'coefficient of parallax' of the body in question, and will be discussed later on. All astronomical observations, being reduced, when necessary (by the application of

the parallax coefficient), to the centre of the earth, and being expressed in angular measure, the heavenly body may be considered, as far as geometry is concerned, as moving on a globe as small as an ordinary celestial globe, an instrument which will be found useful in elucidating astronomical problems, especially in the improved form by Lieut. English.

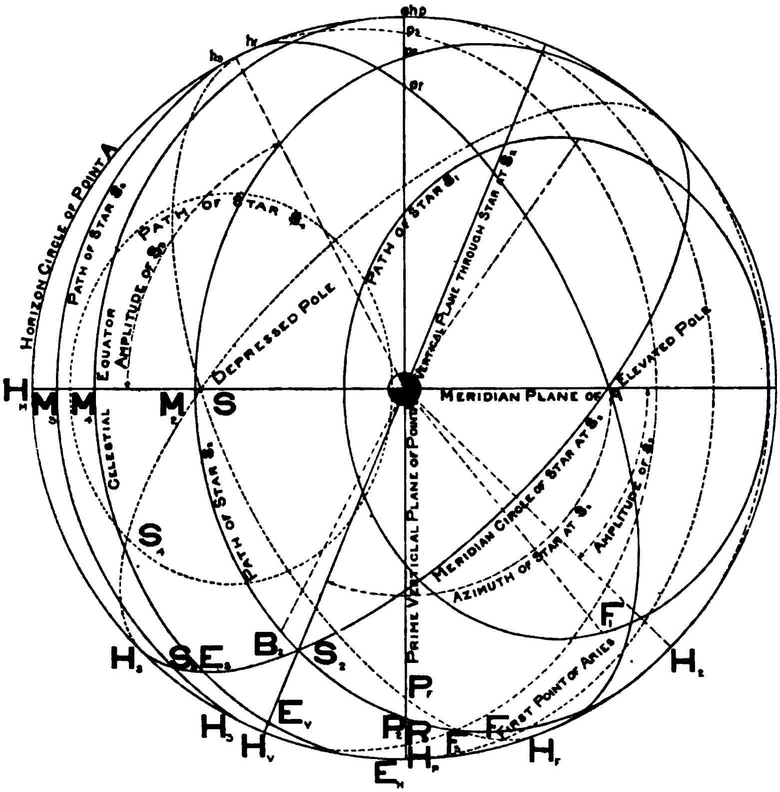


FIG. B.—PLAN ON HORIZON PLANE OF POINT A.

Description of the Orthographic Projections of the Celestial Sphere here given.

Figs. A to E are orthographic projections of the celestial sphere made in five different directions, intended to illustrate the meanings of the terms hereafter defined.

The small globe at the middle of each figure is supposed to represent the earth. All the terms refer to a point of observation A on the earth's surface, and the different views have been derived from one another by projection, so that the same letter denotes the same point throughout. The figures state the views they represent in each case.

The paths of four stars, S_1, S_2, S_3, S_4, are shown. The descriptive terms written on the figures refer in general to the star S_2. The following system of lettering has been adopted:

The celestial poles are lettered N and S, the zenith and nadir Z and D respectively, and the first point of Aries F.

Each circle has a leading letter belonging to it, with which all points on that circle are marked.

Thus E is the letter for the equator, H for the horizon, M for the meridian, P for the prime vertical of the point A, F for the meridian of the first point of Aries, S for the meridian, and V for the vertical circle through the star when at the point S_2 of its path.

The intersection of any two of these circles is lettered with both letters belonging to the two circles. Thus EH is the intersection of the equatorial and horizon circles, EM shows where the equator meets the meridian, M_1 is the intersection of the meridian of A with the path of the star S_1, and so on.

As each pair of circles intersect at two points, one on the upper and the other on the lower half of the sphere, those on the front half on fig. A are referenced

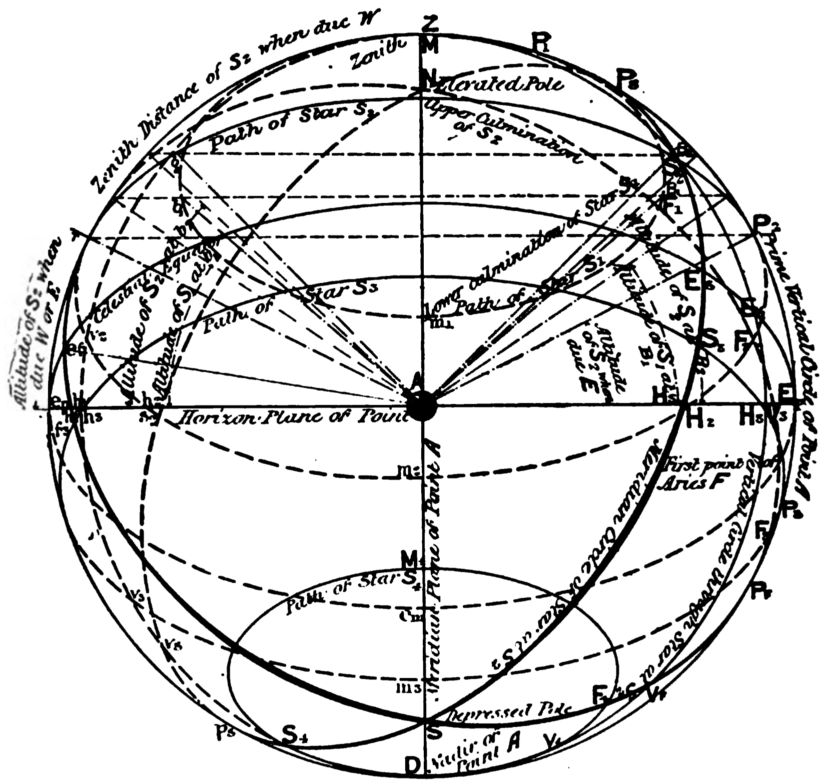


FIG. C.—ELEVATION ON PRIME VERTICAL OF POINT A.

with capital letters, whilst those on the back half are referenced with the corresponding small letters. Thus EH and e_h are the two points of intersection of the equator and horizon. The same letters are then applied to the same points on the other views irrespective of whether they are on the visible or hidden half of the sphere.

Positions of Heavenly Bodies, and how recorded.

The relative positions of heavenly bodies in stellar space, are all mapped and recorded with reference to one 'line' and two 'points.' The 'line' is the 'polar axis' of the earth, prolonged, about which the heavenly bodies appear to revolve daily, and about which the earth does actually revolve. One of the 'points of reference' is the centre of the earth. The second is any 'fixed star,' selected as a point of reference.

Let E, fig. 53 (p. 123), be the centre of the earth, P E P_1 its 'polar axis.' And let R be the 'star of reference.' Then, if we know the angle P E S,

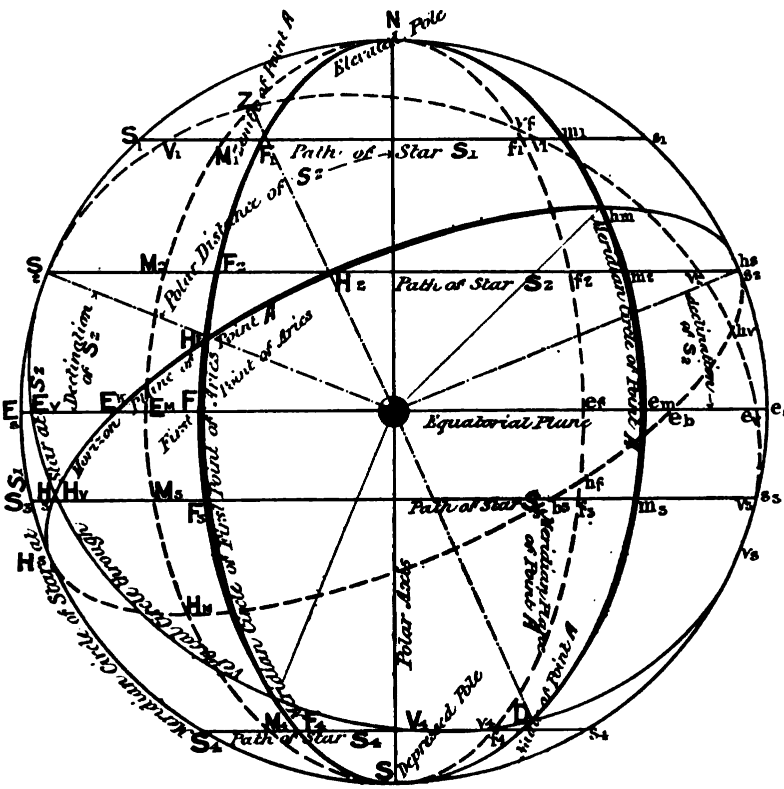


FIG. D.—ELEVATION ON MERIDIAN PLANE OF STAR AT S_2 WITH POLAR AXIS VERTICAL.

included between the line P E P_1 and the line S E drawn from the star S to E the centre of the earth, the position of the star in one direction is known. Further, if the angle Q E S_1, between two planes passing respectively through the 'polar axis' P E P and the stars R and S (as measured in a plane perpendicular to the polar axis P E P_1) is also known, then the position of the star S as observed from the earth's centre is absolutely determined, with reference to the earth's centre, the 'earth's axis,' and the selected 'star of reference.'

The Equatorial Plane or Celestial Equator, Polar Distance, and Declination.

It is the practice of astronomers to record in the tables, ***not*** the angle between a ray, drawn from the star S to the centre of the earth, and the 'polar axis' (this being termed the 'polar distance' of the star), but the complement of this angle, that is to say the angle between the ray S E and a plane through E the centre of the earth (termed the plane of the equator) perpendicular to the axis P E P_1.

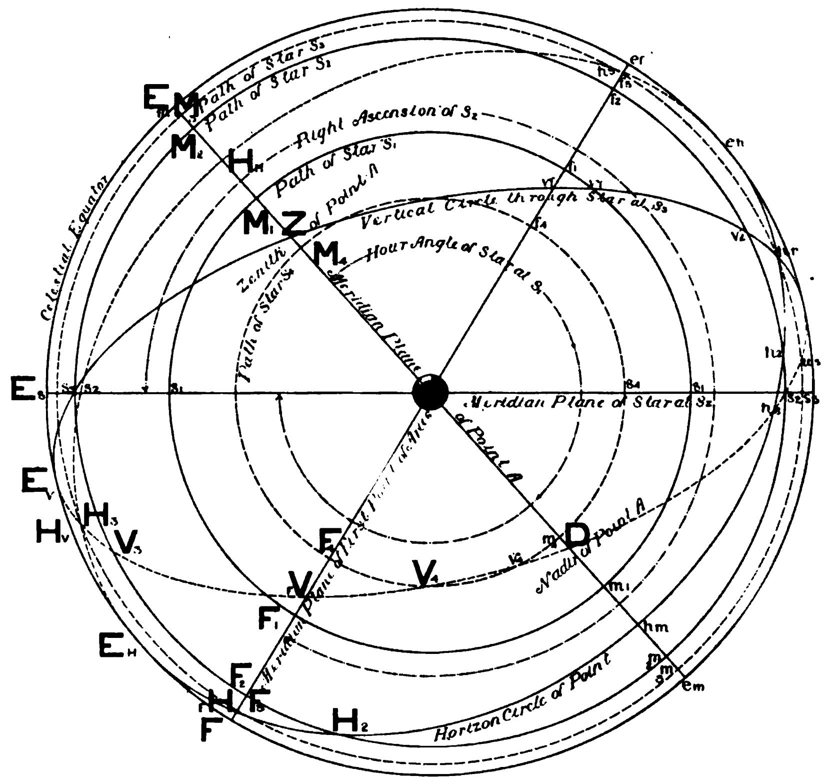


FIG. E.—PLAN ON PLANE OF CELESTIAL EQUATOR.

Declination.

They record in fact the angle S_1 E S, which is called the 'declination' of the star S. The 'plane of reference' for 'declination,' therefore, is the 'earth's equatorial plane,' produced into space. 'declination' is recorded from 0° to 90°, and called 'north' or 'south,' according as the star in question is between the 'equator,' and the 'north' or 'south celestial pole,' respectively. From a geometrical point of view, therefore, the 'declination' of a star coincides exactly with the 'geocentric latitude' of a point on the earth, on the assumption that the earth be a sphere. As the whole radius of the earth is an absolutely negligible quantity, as regards the fixed stars, much more therefore must its slight deviation from the spherical form be, with regard to any heavenly body whatsoever.

Polar Distance. The complement of the 'declination' (i.e. $90° -$ dec.) is the 'polar distance' when the star is on the same side of the equator as the 'elevated pole.' When it is on the opposite side, the 'polar distance' is found by *adding* $90°$ to the declination.

The 'polar distance' is thus always measured from the 'elevated pole.'

Right Ascension. To complete the determination of the star's position, a second angular measurement in a plane perpendicular to the 'earth's polar axis' is required, or in other words the angle between two planes containing, one, the 'polar axis' and the 'base' or 'reference star,' the other, the 'polar axis' and the 'star to be fixed,' and is measured on the plane of the equator, or on any plane parallel thereto. This angle is called the 'right ascension' of the star (*vide* fig. E) and corresponds, geometrically, with the 'longitude' of a place on the 'earth's surface.' The 'plane' or 'meridian' passing through the 'star of reference,' corresponds with the 'meridian of Greenwich,' and the 'plane' passing through the star corresponds with the 'meridian' of some other point on the earth's surface. The notation of angles of 'right ascension' differs from that adopted in the case of 'terrestrial longitude.' 'Right ascension' is reckoned from west to east, from zero to $360°$ (four right angles), whereas 'terrestrial longitude' is reckoned (very inconveniently) from zero to $180°$ eastwards and westwards respectively. 'Right ascension' is denoted by the symbol Æ, and like 'terrestrial longitude' is often expressed in hours, minutes, and seconds, or as subdivisions of four right angles.

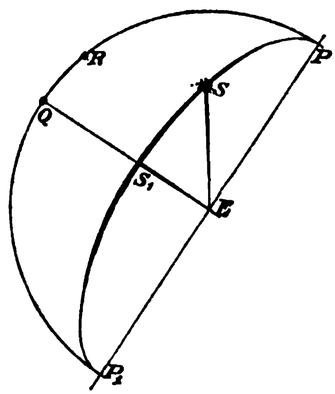


FIG. 53.

First Point of Aries. The 'base star' or 'star of reference' above alluded to is called the 'first point of Aries,' and is denoted by the symbol γ (gamma) being the symbol representing a certain constellation called 'Aries,' or 'the Ram.'

The 'first point of Aries' is indeed an imaginary star, being a certain point in the heavens, with regard to the celestial bodies, determined, and agreed upon by astronomers, as the 'point of reference' or zero of 'right ascension.' For the moment, we must assume that the 'first point of Aries' is marked by an '*actual, fixed* star.'

Celestial Lines and Planes, peculiar to a given place on the Earth's Surface. There are certain 'lines' and 'planes' which are peculiar to, and fixed as regards any one place on the earth's surface.

Polar and Celestial Axes. As we have seen, the 'polar axis' of the earth, prolonged, forms the 'celestial axis' of the stellar sphere.

Terrestrial and Celestial Equators.

The earth's 'equatorial plane,' produced, is the 'celestial equatorial plane' cutting the 'stellar sphere' in a great circle, called the 'celestial equator.' These 'lines' and 'planes' are common to all places on the earth's surface.

Meridian Plane.

There are other planes, which are peculiar to any one spot on the earth's surface, and which remain fixed, with regard to it. The first of these is the 'meridian plane' of the place. This is the plane passing through the 'polar axis' of the earth, and the place of observation, supposed to be prolonged indefinitely so as to cut the 'stellar sphere,' in a 'great circle,' called the 'celestial meridian' of the place.

Zenith and Nadir Points.

The 'zenith' is the point (Z in figs. A to E) in which the direction of a plumb-line, suspended at any point, and prolonged *upwards*, meets the 'stellar sphere.' The 'nadir' (D), is the point in which the direction of the plumb-line, prolonged *downwards*, meets the 'celestial sphere.' The small distance between the intersection of the plumb-line, produced, with the earth's 'polar axis,' and its 'centre,' is absolutely negligible as regards any heavenly body. The line, therefore, joining the zenith and the nadir, passes through the centre of the earth.

Rational, Celestial and Sensible Horizon Planes.

The plane passing through the centre of the earth, perpendicular to the direction of the plumb-line, or the line joining the zenith and nadir of the place, is called the 'rational horizon' of the place, and when produced so as to cut the 'stellar sphere' in a 'great circle' it is called the 'celestial horizon' of that place. It is parallel to the plane of the 'sensible horizon,' which, in the Chapter on the 'Figure of the Earth,' has been defined as 'a plane touching the earth's geodetic surface, at the point of observation, perpendicular to the direction of the plumb-line.' The distance between the 'rational,' and the 'sensible horizon plane' is negligible as regards the fixed stars. In all operations concerning the sun, moon, and planets, bodies that are so near to the earth as to make the difference between the 'rational' and the 'sensible horizon planes' appreciable, one of the first steps in all computations is to apply a correction called the 'correction for parallax' in order to reduce the data, observed on the earth's surface, to those which would have been observed, from the centre of the earth.

Prime Vertical Plane.

The 'prime vertical' is a subsidiary plane, passing through the 'zenith,' the 'earth's centre,' and the 'nadir,' and is perpendicular to the 'plane of the meridian' of the place. It is, therefore, perpendicular to the 'horizon plane.' The 'prime vertical,' moreover, cuts the 'stellar sphere' in a 'great circle.'

Apparent Diurnal Movement of the Fixed Stars.

The 'fixed stars' appear to revolve about the earth, each in a plane passing through the star, perpendicular to the 'polar axis,' and therefore cutting the 'celestial sphere' in small circles (unless with dec. zero) exactly similar to the 'parallels of latitude' on a terrestrial globe.

The exact apparent motions, as regards the observer and the cardinal points, differ materially with the 'latitude' of the place of observation.

[*Note.*—In what follows, the star will be said to be 'visible' when it is situated between the zenith and the horizon or 'above the horizon' irrespective of the fact that, owing to daylight, the star is actually invisible to the unaided eye. The star is there, all the same, but its light is overpowered by the brilliancy of that of the sun. With a telescope of moderate power, several large stars may be seen even in very bright daylight. The star may, therefore, be said to be *geometrically* visible whenever it is above the horizon.]

To an observer at either of the poles, one half of the stars of the universe are always visible in the sense indicated and would appear to move in circles, with the 'zenith' (which in this case coincides with the elevated 'celestial pole') as a common centre, each moving always at the same height above the horizon. The stars in the opposite hemisphere are never visible.

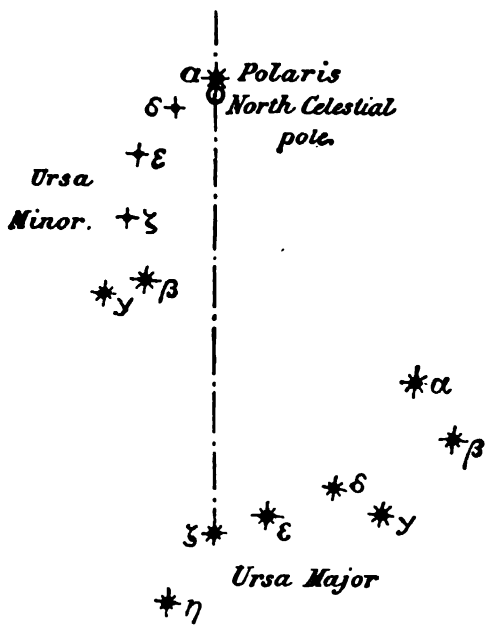


FIG. 54.

To an observer at the equator all stars in the firmament are visible during some part of the day. They appear to revolve about a horizontal 'axis' whose direction is due north and south, and which is neither more or less than the earth's 'polar axis,' and each star will be visible for half the day.

To an observer at an intermediate latitude, a certain number of stars are never visible, never coming above the horizon, others are visible during a certain part of every day, whilst those in a certain zone of the 'celestial sphere' are always visible. For example, the brilliant constellations of the 'Great' and 'Little Bear' (fig. 54) are always visible on any clear night, in the latitude of any place in England. Thus S_1 (figs. A to E) is always visible from A. S_2 and S_3 are visible during part of the day, while they are above the horizon, and S_4 is never visible.

The heavens in the immediate vicinity of the 'south celestial pole' are singularly devoid of brilliant stars. There is indeed a small star, almost exactly in the 'south celestial pole,' but it is so inconspicuous as to be of little practical use. It cannot compare with the 'pole star,' which is distant about $1\frac{1}{4}$ degrees from the 'north celestial pole.'

Circumpolar Stars. Stars which never descend below the horizon are called 'circumpolar stars.' They comprise all stars having declinations '*of the same name but greater than the co-latitude of the place of observation.*' Thus, in latitude 53° N. all stars having a 'north declination' greater than $90^\circ - 53^\circ = 37^\circ$ are circumpolar stars. For example, in figs. A to E, the star S_1 is a 'circumpolar star,' A being the point of observation.

Stars which are Never Visible. Stars having a 'declination' of opposite name to that of the 'latitude' of the place of observation, and greater than its co-latitude, never appear above the horizon. Thus the star S_4 never appears above the horizon of A.

Altitude, Zenith Distance. The 'altitude' (*vide* fig. A), of a star, is the angle subtended at the centre of the earth between the star and the plane of the 'celestial' or 'rational horizon,' measured in a plane passing through the zenith, the centre of the earth, and the nadir. The 'zenith distance' of a star is the *complement* of the altitude. It may also be defined as the angle subtended, at the centre of the earth, between the star, and the direction of the plumb-line, suspended at the place of observation.

Azimuth and Amplitude. The 'azimuth' of a star (*vide* fig. B) is the angle subtended between the plane of the meridian of the place, and the plane passing through the 'zenith,' the star, and the nadir, and therefore perpendicular to the 'horizon plane,' that is to say, the plane from which 'altitude' is measured.

The 'azimuth' of a star, when it is on the 'horizon plane,' that is to say, when its altitude is zero, or zenith distance 90°, is called its 'amplitude.'

'Azimuth' is noted sometimes on the 'whole circle,' but more commonly on the 'four quadrant' system.

Transit or Culmination. At the instant at which a star passes the 'meridian' of the place of observation, it is said to 'transit' or 'culminate,' or reach its 'culmination,' for at this instant the star's altitude is greatest.

The passage of a star over the 'meridian' of a place is called its 'transit.' Circumpolar stars transit twice in the day *above the horizon*, once *above* and once *below* the 'elevated pole,' that *above* the pole being called the 'upper culmination,' that *below* the 'lower culmination.' All stars pass the 'meridian' of the place of observation twice in the day, but the lower culminations or transits of other than circumpolar stars are not visible.

Hour Angle. The 'hour angle' of a star (*vide* fig. E) is the angle between the 'plane of the meridian' of the place of observation, and a plane passing through the star, the celestial poles, and the centre of the earth, measured in the 'plane of the equator,' or in any plane parallel thereto. Hour angles are sometimes expressed in degrees, minutes and seconds, reckoned on

the 'whole circle' system, from zero when on the 'meridian' or at the instant of 'upper transit' round to 180° when at the 'lower transit,' and so on to the 'meridian' again. More often, 'hour angles' are noted on the 'four quadrant' system, being the smallest angles from the nearest 'meridian,' and stating whether the star is *east* or *west* of the same.

Very frequently, like terrestrial longitudes, the 'hour angle' is expressed in hours, minutes, and seconds, for in addition to meaning a geometrical angle, 'hour angle' represents the actual time elapsed at the moment of observation, from the instant of the stars *preceding* 'upper transit' over the 'meridian,' the time which elapses between two consecutive 'upper transits' being reckoned as twenty-four hours.

Solar and Sidereal Time. The ordinary civil time of day at any place is simply the *mean* sun's 'hour angle' at the given instant (less twelve hours if over that interval). When the sun is on the 'meridian' it is noon. At 3 P.M. the sun's 'hour angle' is $\frac{3}{24}$ of four right-angles, or 45°, and so on. The instant at which the 'first point of Aries' is on the 'meridian' of any place, is called 'sidereal noon,' and the 'sidereal time,' at any other instant, is the 'hour angle' of the 'first point of Aries' at that instant.

Relation between Altitude, Azimuth, and Hour Angle. An inspection of figs. A to E, or better, of a celestial globe, makes it evident that the following relations obtain between 'altitudes,' 'azimuths,' and 'hour angles.' When the 'altitude' of a star after 'culmination' is equal to that which it had at some time previous to its 'transit,' then, the 'hour angle' of the star when *rising* was equal to 360° less its 'hour angle' when *falling* (whole circle reading). So also with 'azimuths.' The 'azimuth' of a star, when it has gained a certain altitude before 'culmination,' when *rising*, is equal to 360° less its 'azimuth' when it has the same 'altitude' when *falling*. This is reckoning by the whole circle system. Starting with zero at the point of 'upper culmination,' on the four quadrant notation, the 'azimuths' corresponding to 'equal altitudes' are numerically *identical*, one being *east* and the other *west*.

Elongation. The instant at which 'circumpolar stars' attain their greatest distance from the 'meridian,' east or west thereof, is called the instant of their 'elongation.' It is evident that the angles which the star's 'azimuth plane' make with the 'meridian,' at the instants of 'elongation,' are equal.

Formulæ for Predicting a Star's Position when on the P.V., at Elongation, and at its Amplitude. When a star is on the prime vertical, at elongation, or at its amplitude, the astronomical triangle is right-angled as follows—

Azimuth angle = 90° when on the P.V.
(Angle at star) Parallactic „ = 90° when at elongation.
Zenith distance = 90° when at its amplitude.

Now, from Napier's rules for circular parts we have

Sine of middle part = product of tangents of adjacent parts.
= product of cosines of the opposite parts.

Now, if C be the right angle (at the star) as when the star is at its 'elongation,' then

$$\sin (90 - B) = \tan (90 - c) \tan a$$

(*Vide* Appendix A for letters.)

or

Similarly

$$\left.\begin{aligned} \cos t &= \tan \phi \cot \delta \\ \sin A &= \cos \delta \sec \phi \\ \sin h &= \sin \phi \operatorname{cosec} \delta \end{aligned}\right\} \text{giving time, azimuth, and altitude.}$$

Again, if A be the right angle, as when the star is on the 'P.V.,' then

$$\left.\begin{aligned} \cos t &= \tan \delta \cos \phi \\ \cos z &= \sin \delta \operatorname{cosec} \phi \end{aligned}\right\} \text{giving time, and zenith distance.}$$

Lastly, if the side '*b*' be the right angle, as when the star is at its amplitude, then

$$\left.\begin{aligned} \cos A &= \sec \phi \sin \delta \\ \cos t &= \tan \phi \tan \delta \end{aligned}\right\} \text{giving azimuth, and time.}$$

Rough Determination of the Meridian, or true North and South Line.

The above principles suggest two ways of determining the *true* 'north-and-south line,' without any knowledge of the star's position in the heavens.

Suppose that a theodolite be set up and adjusted, with plates clamped to 'zero,' and that by means of the compass, the line of collimation of the telescope be made approximately true north and south. Clamping the lower limb, let some rising star be intersected by the cross-hairs. Now read the horizontal angle or the star's bearing with the approximate north. Keeping the vertical arc firmly clamped, wait till the star, after culmination, has reached about the same but rather greater altitude, than it had when observed rising. Then, turning the upper plate, direct the telescope towards the star, always leaving the 'vertical' arc firmly clamped. As soon as the star appears in the field of view, intersect it with the vertical wire, clamp and keep the star intersected by the vertical wire, by gently turning the tangent-screw of the horizontal plate, until the star crosses the 'horizontal' wire, being at the same time kept on the vertical wire. At this instant the star has the same altitude as at the first observation. Stop moving the upper horizontal plate, and read the horizontal limb again. Now, if the instrument has been set exactly north and south ($0°$ being north), the first reading of the horizontal limb will be equal to the difference between $360°$ and the second reading.

If these two angles be not exactly equal, then the line of collimation of the telescope, when the plates are set at zero, is out of the true plane of the meridian by half the difference between the angles obtained. Thus, in fig. 55 let S and S_1 be the positions of the star in 'azimuth' before, and after culmination, respectively, then if O C be the direction of due north O C will bisect the angle

S O S_1. Let O D represent the direction of the telescope, and let D O S = θ be the first reading to the star rising, and D O $S_1 \times \theta_1$ the second reading, then S O $S_1 = \theta + \theta_1$.

$$\text{Now} \qquad \angle COS = \tfrac{1}{2} \angle SOS_1 = \tfrac{1}{2}(\theta + \theta_1)$$

$$\text{but} \qquad \angle DOS = \theta$$

$$\therefore \qquad \angle COD = \angle COS - \angle DOS = \tfrac{1}{2}(\theta + \theta_1) - \theta$$
$$= \tfrac{1}{2}(\theta_1 - \theta)$$

or half the difference between θ_1 and θ.

By moving the upper limb, in the right direction, through an angle equal to this semi-difference, the telescope will be placed in the true 'plane of the meridian.'

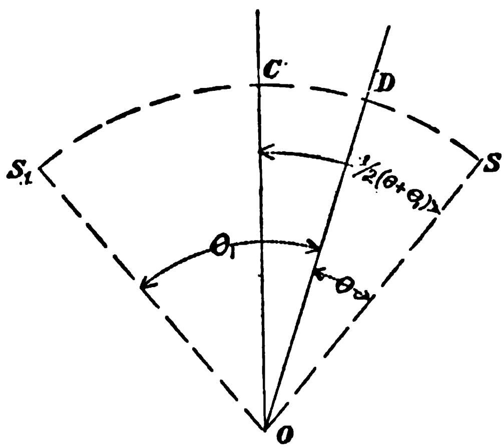


FIG. 55.

At dawn, a signal or rod may be ranged out, due north or south of the point of observation from which angles may be observed to distant points. These angles will be the *true* 'azimuths' of the points.

In a similar way, the *true* north may be found by reading horizontal angles to a circumpolar star, at two successive elongations. Neither of these methods are of much practical use, that by 'equal altitudes,' because of the possibility of losing the second observation by the sky being overcast, that by 'elongations,' because both 'elongations' rarely occur during the hours of darkness. Both methods are however feasible, and, with minor modifications, are actually used.

Rough determinations of Latitude by Culminations of Circumpolar Stars.

The latitudes of places, sufficiently distant from the equator to make a brilliant circumpolar star available for observation, may be obtained, without any knowledge of the position of the star as regards 'declination,' by observing its 'altitudes' at the instants of its 'upper' and 'lower culminations' respectively. It will be obvious, by inspection of fig. A, that the arithmetical mean of the two 'meridian altitudes' of the circumpolar star is

the altitude of the 'celestial pole,' that is to say the 'altitude' of the point in the 'celestial sphere' about which the star appears to revolve. This, by definition, is the 'latitude' of the place of observation. Thus, in fig. A, H*m* O N is the mean between H*m* Om_1 and H*m* O M_1. and gives the latitude. The above method is applicable for the purpose of establishing the 'latitude' of a fixed observatory, because upper culminations could be observed at one time of the year, lower at some other. They are described in this place mainly to illustrate the apparent motions of the fixed stars, and the manner in which their movements are used to determine latitude, azimuth, and time.

The Ephemeris or Nautical Almanac.

When it is desired to obtain either 'latitude,' 'azimuth,' or 'time' by a single observation of a heavenly body, a knowledge of the star's position, as to 'declination' at least, is requisite, and for 'time,' its 'right ascension' must also be known. This is to be obtained by reference to an ephemeris or almanac, the Nautical Almanac for example.

Most of the requisite information is given in other almanacs, such as Whitaker's, which is abridged from the Nautical Almanac.

Description of Methods of taking out Data.

To facilitate the comprehension of the 'Nautical Almanac,' a brief description is desirable of the method used in obtaining the data as to the positions of the fixed stars, sun, moon, and planets, and for investigating their movements and the laws which govern them, and finally, for predicting the positions which they will occupy at future instants of time. Perhaps the simplest method is to assume that the student desires to re-open the whole question and investigate it *de novo*. Possessed of a transit theodolite and a good watch (rated to civil solar time) he has, as far as principle is concerned, all the essentials of an observatory.

First, let a theodolite be set up, levelled, and adjusted, so that the line of collimation is exactly in the plane of the 'meridian,' by methods analogous to those which have been above described.

Determination of Right Ascension.

Having so done, let the instant at which any one 'fixed star' passes the vertical wire of the theodolite, or in other words, its 'time of transit' be noted, night after night by means of the watch. Now if the watch is going properly, or indicating correct civil time, two facts will be observed. Firstly, the time intervals between any two successive transits will always be equal, though *not* twenty-four hours by civil solar time. Secondly, that by correct civil time the 'transit' of any 'fixed star' will take place about four minutes earlier, on each successive night. At this stage of the proceedings, let us suppose that the observer alters the regulator of his time-keeper, and so adjusts its rate that *exactly* twenty-four hours elapse, between two 'superior transits' of some given 'fixed star.' Having so done, his watch will be found to be gaining on a civil time clock keeping good time at the rate of about '*four minutes*' per day, or more nearly '*one day*' in a year.

Next, let us continue to suppose that the 'base star' or 'first point of Aries is a real visible 'fixed star.' [Base or reference stars are used by astronomers, but the measurements, deduced from them, are finally reduced to the imaginary

point in the heavens, called the 'first point of Aries,' whose position in the heavens will be discussed later on.]

Then, at the instant at which this 'base star' or 'first point of Aries,' passes the 'meridian,' as marked by its transit over the centre wire of the telescope, let the hands of the watch be set to 12 h. 0 m. 0 s. Now if the time of transit of any other 'fixed star' be noted by the watch so regulated, and set, the said time of transit will be the 'right ascension' of the said 'star.' [*Note.*—The clocks (S.T. or M.T.) used by astromomers in an observatory, have the hour dials divided into twenty-four hours, the hour hand making one revolution during the day. If a common clock were used, then if more than twelve hours have elapsed between the transit of the 'first point of Aries' and that of the 'star' whose position is to be determined, twelve hours must be added to the time of transit of the star.]

Now, the geometrical determination of the 'right ascension' of a star, considered as the equatorial angle, or fraction of 'four right angles' between the 'first point of Aries' and the 'star' in question, is wholly independent of the rate or error of the time-keeper, provided always that the rate is uniform, and it is therefore only necessary to work out the following proportion sum to find a star's R.A. (or Æ).

As the number of seconds elapsing between two successive transits of the 'base star' or 'first point of Aries,' is to the number of seconds elapsing between the transit of the 'star' and the preceding transit of the 'first point of Aries,' so are four right angles (whether expressed as twenty-four hours, or any other subdivision), *to the right ascension of the star in question.*

It is the practice of astronomers, however, to have a clock of the best possible construction, regulated and set to sidereal time, with the greatest possible accuracy. Such a clock is called a 'sidereal clock,' and (except as to superiority of construction) differs only from an ordinary eight-day clock, in that the hour dial is divided into twenty-four hours instead of only twelve.

At the instant of the star's transit let its 'altitude' (in other words its 'meridian altitude') be observed. The 'latitude' of the observatory being accurately determined, by a method similar to that already described, then, by adding or deducting the star's altitude to or from the co-latitude of the place as shown in fig. A, the declination of the 'star' is determined.

Apparent Motions of Fixed Stars.

By means of observations conducted on these principles and extending over many years, the positions of the fixed stars have been determined. From these it appears that even the so-called 'fixed stars' have motions, some *apparent*, some *real.* The discussion of these 'apparent' or 'real motions,' comes under the province of physical astronomy. Suffice it to say that the laws governing these motions *are known in every case*, so that the position of a 'fixed star' in 'right ascension' and 'declination' can be predicted many years in advance. These predictions are given in the Nautical Almanac, pages 295 to 305, and in greater detail in pages 326 to 455.

In these latter tables the 'right ascensions' of sixteen stars are given daily, and of 392 others at intervals of ten days. The 'right ascension' of the star as

given in the N.A. is the hour shown by the sidereal clock at Greenwich at the instant of the star's 'meridian passage' over the 'centre wire' of the principal 'transit instrument.' As the daily differences are at all times small, the 'right ascension' of a 'fixed star' for any given day, as given at Greenwich, may be taken (for all practical purposes) as the 'right ascension' of that star at any other place, that is to say, the time that the 'sidereal clock' at that place should show at the instant of 'transit' of the given star.

The Apparent Movements of the Sun, Moon, and Planets.

As above described, the apparent positions of the sun, moon, and planets may be determined, day by day, simply by observing their 'times of transit' and 'meridian altitudes.' Let the 'sidereal time' of the 'transit' of the sun's centre be noted on any day, this gives the sun's 'right ascension' at (apparent) noon at the place of observation. By determining the 'altitude' of the sun's centre, its 'declination' for the same instant is determined. By repeating these operations day after day, and year after year, the sun's apparent motion has been determined, and its real motion (or rather that of the earth) ascertained, as well as the laws which govern it. With these data, the sun's apparent place is predicted in the Nautical Almanac for a given instant (noon at Greenwich), for every day in the year. So also for the moon, and planets.

Sun's Apparent Annual Movement.

The attached diagram (fig. F) shows the sun's apparent path across the celestial sphere, and through the fixed stars, for a period of 'one calendar year,' commencing on the 21st March, 1900, and ending on the 21st March, 1901. The centre line denotes the celestial equator, along which the 'right ascensions' of the 'fixed stars' and 'sun' are plotted as abscissæ, numbering from west (right) to east. 'Declinations' are plotted as 'ordinates.' If the diagram were bent into a hoop, with the extremities brought together, it would represent, approximately, a zone of the celestial sphere, as seen by the eye in the centre thereof, and it may be compared with a celestial globe as seen from outside.

Equinoxes.

Twice in the year the 'sun's declination' is zero, once in the spring, and again in the autumn. The instant at which the 'sun's declination' is zero, or in other words, the instant at which its centre is exactly on the equator, is called the 'equinox.' That occurring in spring is called the 'vernal equinox,' that in the autumn the 'autumnal equinox.'

Summer and Winter Solstices.

The maximum 'north declination' is attained in midsummer (in the northern hemisphere), the maximum 'south declination' in midwinter. The instants of 'maximum declinations' north and south are called the 'summer' and 'winter solstices' respectively.

The 'time interval' between two successive 'vernal equinoxes' is that adopted as the 'solar' or 'equatorial year,' and amounts to 365¼ ordinary civil days. (More accurately 365 days 5 hours 48 minutes and 49·7 seconds.)

First Point of Aries.

The point in the heavens (with regard to the 'fixed stars') at which the sun arrives (in R.A.) at the 'vernal equinox,' is the 'first point of Aries,' and this is the starting point of all measurements in R.A. The 'sun's declination' is also zero at that instant, so that at the

The true path of the Sun is shown thus —●— The mean path would be indistinguishable from it on the scale of the figure. The difference between mean and apparent time is shown in figure a. The time from the curve is to be added to the apparent time to find the mean time. i.e. M = A + (±) E

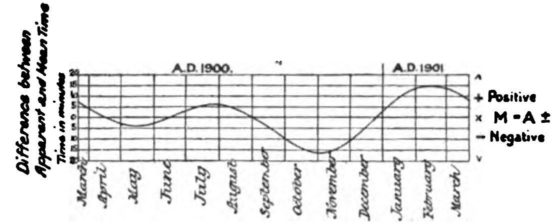


FIG. *a.*

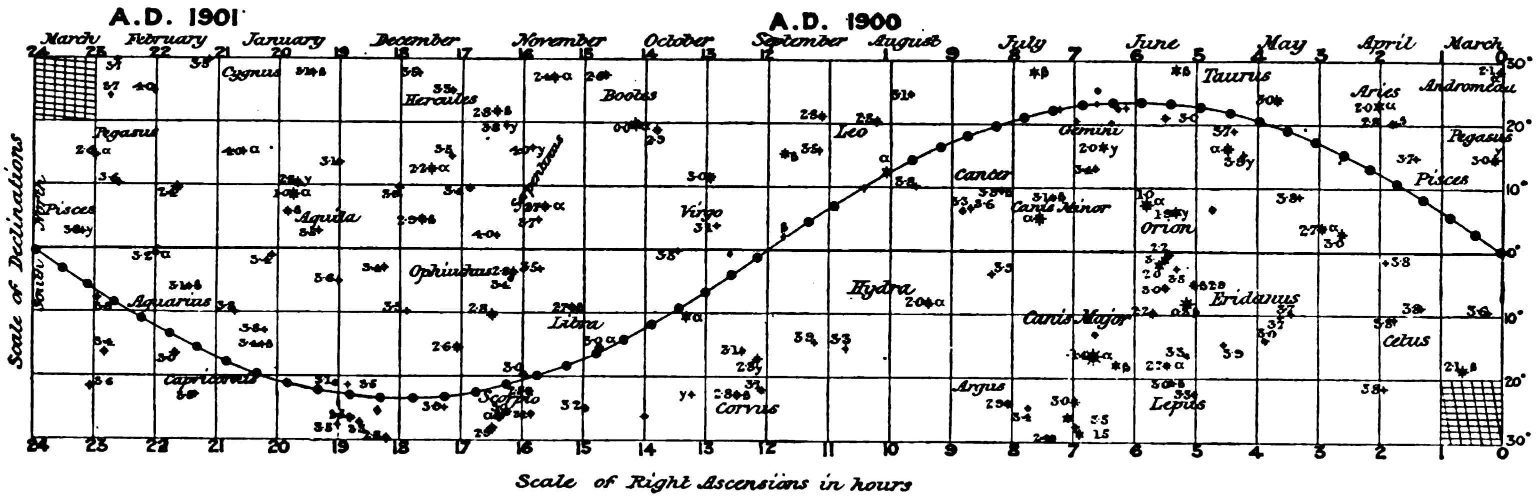


FIG. F.

instant of the 'vernal equinox,' the 'sun's declination' and 'right ascension' are both *zero*, and on the succeeding 'vernal equinox' both again become *zero*.

Equatorial Year.

Reasons for Leap Year.

The 'equatorial year' is equal, as regards time interval, to the 'ordinary calendar year,' but the latter dates from midnight. Again, as the 'true year' contains a fractional number of days, it would be necessary to reset all clocks on 1st January. To obviate this, the year is made of 365 days for three consecutive years, and the approximate fourth part is adjusted, by adding one day to every fourth year (or leap year). This rather over-corrects the error, and therefore a leap year is omitted once in a century.

During the course of the 'equatorial year' the sun makes a complete turn of the 'celestial sphere,' moving from west to east, or in the direction opposite to the apparent daily motion of the 'celestial sphere.' The consequence of this retrograde motion is that the sun, during the course of the year, makes a number of transits *less* by *one*, than does the 'first point of Aries' or any 'fixed star.'

During the year, i.e. the 'interval' between two 'vernal equinoxes,' the sun transits 365 times, and further, attains an hour angle of *about* six hours. The 'first point of Aries' transits 366 times, and also attains a further hour angle of *about* six hours.

Sidereal, and Mean Solar Day

For the above reason, the 'sidereal day,' or the 'time interval' between two successive transits of the 'first point of Aries,' is *shorter* by about four minutes than the 'mean solar day,' which is the 'mean time interval' between two successive 'transits' of the sun over the 'meridian' of the place of observation, during the whole year.

Mean Sun.

The '*mean* interval' is specified, advisedly, because the sun does not move uniformly in 'right ascension,' being sometimes more than a quarter of an hour in advance, at others, as much behind its mean or average position. If therefore, the actual 'transits' of the sun were used to regulate noon, clocks would have to vary their rates continually, to keep pace with the sun. To obviate this inconvenience astronomers have assumed the existence of an *imaginary* (mean) sun, which moves uniformly through the circle of 'right ascension' during the year.

Local Apparent Noon.

Local Apparent Time.

The instant at which the centre of the *real* sun passes the 'meridian' of any place is called 'local apparent noon.' The actual 'hour angle' of the *real* sun, measured from the 'meridian' and expressed in hours, etc., is called 'local apparent time' at the place of observation. This is the time shown by a sun-dial.

Local Mean Noon.

Local Mean Time.

The instant at which the *imaginary* 'mean sun' passes the 'meridian' of any place is called 'local mean noon' at that place, and the 'hour angle' of the 'mean sun' measured from the 'meridian' and expressed in hours, etc., is called local mean time' at that place.

Civil and Astronomical Dates and Hours Compared.

The ordinary civil day and date commence at midnight, that is, 12 mean solar hours before 'mean noon' at the place of observation. The hours are numbered from 0 (midnight) to 12 noon, and again on to midnight, the hours between midnight and noon being called 'ante-meridian' or A.M. and between noon and midnight 'post-meridian' or P.M.

In the astronomical notation, the day and date commence at 'noon,' and the hours are numbered from 0 to 24. This notation is always adopted in the Nautical Almanac.

Conversion of Civil Date and Hour into Astronomical Reckoning.

The first step towards obtaining information as to the position of any heavenly body from the Almanac, is to convert the *civil* 'time' into the corresponding *astronomical* 'time and date.'

The rule is as follows:—

For hours A.M. *(civil time) add* 12 *to the hour, and deduct* one *from the date. The result will be the astronomical 'hour and date.' For hours* P.M. *the hour and date are identical in both reckonings.*

Example.—

6 h. 30 m. A.M. on 5th February (civil reckoning), is equivalent to 6 h. 30 m. + 12 h. = 18 h. 30 m. on 4th February (astronomical reckoning).

On the other hand, 6 h. 30 h. P.M. on 5th February (civil reckoning), is simply 6 h. 30 m. (astronomical reckoning).

Conversely, to reduce 'astronomical' to 'civil reckoning.'

If the 'astronomical hour' be greater than twelve, take twelve hours from it. Mark the remainder A.M. *and add 'one' day to the date.*

When the 'astronomical hour' is less than twelve, then the two reckonings coincide, and it is only necessary to mark the hour P.M.

Example.—

Fourteen hours on 12th June (astronomical reckoning), is equivalent to 14 − 12 = 2 A.M. on 13th June (civil reckoning).

Three hours 'astronomical reckoning' on 11th August, is equivalent to 3 P.M. 'civil reckoning' on the same date.

Description of the Nautical Almanac.

The first two pages for each month in the Nautical Almanac give the sun's 'right ascension' and 'declination' at 'apparent' and 'mean noon' at Greenwich, respectively.

The 'right ascension' at 'apparent noon,' is mainly of use in checking the sidereal clock, by means of transits of the sun. It is the hour that the Greenwich sidereal clock should show at the instant of the transit of the sun's centre. By adding the appropriate difference, which is given, the sun's 'right ascension' at any other hour may be computed. For stations other than Greenwich, first deduct or add the difference due to longitude, east or west, that is to say, ascertain the Greenwich date and hour corresponding to the instant of apparent noon at the station in question, then correct the 'right ascension' of that date to the corresponding Greenwich instant, and the 'right ascension' required will be obtained. This will be the correct sidereal time at the instant of the sun's transit at the station of observation.

The sun's 'declination' at 'Greenwich apparent noon,' is the declination which the sun has, at the instant of its transit at Greenwich, as determined by a 'meridian altitude.' 'Declination' at 'apparent noon,' is useful in determining the 'latitude' of a place by a 'meridian altitude' of the sun, as this 'altitude' takes place at 'local apparent noon.' When the place of observation is not on the 'meridian' of Greenwich, the 'declination' must be corrected by adding the proper hourly difference, multiplied by the number of hours elasped (due to longitude east or west) between 'Greenwich noon' and the 'Greenwich instant' at which the observation was made.

The sun's 'right ascension' and 'declination' are also given for 'Greenwich mean noon,' that is to say the instant at which the *imaginary* mean sun passes the 'meridian' of the place.

Both 'declination' and 'right ascension' require reduction to the meridians of places not in the 'longitude' of Greenwich, in the manner that has been described in the case of 'apparent noon.' Lastly, on page II. of each month there is an important column, headed 'sidereal time' at 'mean noon.' This is the hour that the Greenwich sidereal clock should show at the instant of 'Greenwich mean noon.' It may also be described as the 'right ascension' of the *mean sun* at 'Greenwich mean noon.' It differs from the 'right ascension' of the *real* sun, in that daily differences are exactly equal, whereas those of the latter vary. The 'sidereal time' at 'mean noon' also requires correction to meridians other than that of Greenwich, the method of effecting which will be described later on.

Mean Time at Sidereal Noon.

On page III. of each month there is a column headed 'mean time' at 'sidereal noon.' Now 'sidereal noon' is the instant at which the 'first point of Aries' passes the 'meridian,' and the hours given are those which the 'Greenwich mean time clock' should show daily when it is 'sidereal noon at Greenwich.' This column is used for the conversion of 'sidereal' into 'mean time' (*vide* page 140).

Equation of Time.

The difference between 'mean' and 'apparent time' is given on pages I. and II. each month, in a column headed 'equation of time.' These differences vary in amount, and are sometimes to be *added to* and at others to be *deducted from* 'apparent time.' They must be corrected for 'longitude' and 'hour,' but the correction is small, as seen from the column of hourly variations. The 'equation of time' is also the correction to be applied to the time indicated by a sun-dial to reduce its readings to 'mean time.' Mean and apparent times are, therefore, connected by the equation M.T. $=$ A.T. $\pm$ E, where E is the equation of time.

Semi-diameter.

The diameter of the sun's disc is considerable, subtending nearly half a degree. As there is no central point, it is usual to observe 'altitudes' to one or other of the edges of the disc, which are termed the 'upper and lower limbs' respectively. If only one 'limb' is observed, then, in order to obtain the true 'altitude' of the 'sun's centre,' its semi-diameter must be *added* or *subtracted* according as the lower or upper limb is observed. The sun's semi-diameter is given daily on page II. of each month.

Sidereal Time of Sun's Semi-diameter Passing Meridian.

For similar reasons, it is customary, when observing a transit of the sun, to note the instant of the contact of the *leading* and *following* 'limbs' with the 'vertical' wire (or wires) of the telescope, and to take the 'mean of the times' as the 'instant of transit' of the 'sun's centre.' It is possible that owing to driving clouds, one or other of the observations may be lost. The time of transit of the semi-diameter being known, such a partial observation can be utilised.

Sidereal Time, Right Ascension, and Hour-Angle.

'Sidereal time' at any instant is the 'time interval' in sidereal hours, minutes, and seconds, which has elapsed since the transit of the 'first point of Aries' over the 'meridian' of the place. It is also, geometrically, the 'hour angle' of the 'first point of Aries,' at that instant, measured from its upper transit over the 'meridian' of the place.

The 'right ascension' of a star is the 'sidereal interval' elapsing between the 'upper transit' of the 'first point of Aries,' and the 'upper transit' of the 'star.' Geometrically, it is the angle in the equatorial plane, between the 'star' and the 'first point of Aries.'

Lastly, the 'hour angle' of a star at a given instant is the 'sidereal time, elapsing between the instant of its 'upper transit' and the given instant. Geometrically, it is the angle between the 'meridian' of the place, and the 'star' at the given instant of time measured in the plane of the 'equator,' the notation being on the 'whole circle system.' Therefore, if the 'hour angle' of a star at any instant of time be obtained, the 'sidereal time' of that instant is obtained by *adding the star's 'hour angle' to its 'right ascension,' and, if the sum exceeds twenty-four hours, deducting that amount.*

Conversion of Sidereal, into Mean Time Intervals.

'Intervals of time' expressed in 'sidereal hours, etc.,' may be converted into 'mean solar time intervals,' and *vice versa*, by the following equivalents.

Since, in the time the earth has described its orbit round the sun, the latter has made one 'transit' less over any 'meridian' than the first 'point of Aries'—

$$\therefore\ 366{\cdot}2422 \text{ sidereal days} = 365{\cdot}2422 \text{ mean solar days.}$$

$$\therefore\ 1 \text{ ,, day} = 0{\cdot}99727 \text{ ,, day.}$$

or

$$24\text{ h. S.T.} = 23\text{ h. } 56\text{ m. } 4{\cdot}091\text{ s. of M.T.}$$

$$\therefore\ 1\text{ h. S.T.} = 1\text{ h. M.T.} - 9{\cdot}8296\text{ s. of M.T.} \qquad (1)$$

and

$$1\text{ h. M.T.} = 1\text{ h. S.T.} + 9{\cdot}8565\text{ s. of S.T.} \qquad (2)$$

Equation (1) shows the *retardation* of M.T. on S.T.; and (2) the *acceleration* of S.T. on M.T. The Nautical Almanac gives a table for this conversion.

Reduction of Right Ascensions and Declinations, to Greenwich Hour and Date.

Data as to the 'right ascensions' and 'declinations' of the sun, moon, and planets, are given in the Nautical Almanac (in future called the N.A.) for certain instants of Greenwich time, namely, 'apparent noon' and 'mean noon,' and by means of the 'differences' also given, the required data may be corrected to any subsequent 'Greenwich time.' To correct such data for the time of observation at any given place, it is necessary to reduce this time of observation to its corresponding 'Greenwich instant' or 'Greenwich hour and date' by applying the correction due to longitude, east or west, at the rate of one hour for 15°.

If the 'longitude' be 'east,' then *deduct* the difference of 'longitude' in hours from the 'local time,' and the *difference* will be the corresponding 'Greenwich instant.' If, however the 'longitude' be greater than the 'local time,' *add* 24 hours to the 'local time' and *deduct* one day from the date. The result will be the Greenwich hour and date.

If the 'longitude' be 'west,' then add the difference of 'longitude' to the 'local time.' If the sum exceeds 24 hours, *deduct* 24 hours and *add* one day to the date.

As the surveyor will usually be provided with a watch indicating ordinary 'local mean time,' it will be convenient to perform the two reductions simultaneously.*

Examples.—

(1) An observation of the sun was made at 9 h. 15 m. A.M. on the 6th June, in 'longitude' 10 h. 30 m. east. Find the Greenwich (instant) hour and date.

Here, 21 h. 15 m. on the 5th June is the corresponding astronomical time and date. Deducting for longitude east 10 h. 30 m., we have 10 h. 45 m. on the 5th June the corresponding Greenwich hour and date.

(2) At 3 P.M. on the 3rd July in 'longitude' 11 h. 15 m. east. Required the Greenwich hour and date.

Here, 3 hours on the 3rd July is the corresponding astronomical time and date. Adding 24 hours and deducting for longitude east 11 h. 15 m., we have 15 h. 45 m. on the 2nd July the corresponding Greenwich hour and date.

(3) At 11.18 A.M. on the 12th August in 'longitude' 6 h. 30 m. west. Required the Greenwich time and date.

Here, 23 h. 18 m. on 11th August is the corresponding astronomical time and date. Adding for 'longitude' west 6 h. 30 m. we have 29 h. 48 m., then deducting 24 h. and adding one day to date, 5 h. 48 m. on the 12th August is the Greenwich time and date, at the same instant.

(4) At 8 A.M. on the 16th May in 'longitude' 165° west. Required the corresponding Greenwich time and date.

	h.	m.	s.	
8 A.M. on the 16th, by astronomical reckoning =	20	0	0	on the 15th.
Add for longitude west 165° =	11	0	0	
	31	0	0	
Deduct 24 h. . . .	24	0	0	
We have Greenwich time . . . =	7	0	0	on the 16th May.

* The mariners' saying is :—

Longitude west.
Greenwich best.

Longitude east.
Greenwich least.

The following abbreviations will be used (*vide* also Appendix A, page 280) :—

N.A.	Nautical Almanac.	L.M.T.	Local mean time instant.
G.	Greenwich.	G.A.T.	Greenwich apparent time.
G.M.T.	Greenwich mean time instant.	L.A.T.	Local apparent time.

Conversion of Mean to Apparent Time, and vice versa.

The conversion of 'mean' to 'apparent time' and *vice versa* is effected by the use of the formula M.T. = A.T. ± E, given above under 'Equation of Time,' where M.T. denotes 'mean time,' and A.T. 'apparent time,' ± E signifying the 'equation of time' to be 'added to' or 'deducted from' the latter. The 'equation of time' is taken from pp. I. and II. of each month in the N.A. at 'apparent' or 'mean noon,' and calculated for the 'meridian' of Greenwich. The 'hourly variation' of E is also given, so that its value can be reduced to the corresponding Greenwich instant, when at a place of different longitude.

The following examples show the calculations necessary.

(1) Find the 'local mean time' of 'apparent noon' on the 16th November, 1879, in 'longitude' 65° 10′ east = 4 h. 20 m. 40 s. = 4·34 h.

At G.A.N. on above date, E = − 15 m. 6·85 s.

Hourly variation = − 0·464 s.

∴ Change of E for longitude = − 0·464s. × 4·34 = − 2·02

and E at local apparent noon = − 15 m. 8·87 s.

Now

Greenwich mean time of apparent noon = 0 h. 0 m. 0 s.

Hence, deducting E, we have

Local mean time of apparent noon = 23 h. 44 m. 51·13 s.

on the 15th November (astronomical reckoning).

(2) In longitude 65° 10′ given 8 h. 54 m. 10 s. A.M. (civil reckoning) on the 16th May, 1879, 'local mean time,' to find the 'L.A.T.'

	h.	m.	s.	
Here L.M.T. (civil) =	8	54	10	A.M. on the 16th.
Add 12 h. . .	12	0	0	
	20	54	10	on the 15th.
Corrections for long. E −	4	20	40	
G.M.T. . . =	16	33	30	(astronomical reckoning).

From Tables, E = − 15 m. 17·47 s.

Variation per hour = − 0·441 s. (found by interpolation, for the middle instant between Greenwich noon and 16 h. 33 m. 30 s.).

L.A.T. Local apparent time, *vide* Appendix A.

$$\therefore\quad -0{\cdot}441 \times 16{\cdot}57 = +\overset{\text{h.}}{}\ \overset{\text{m.}}{0}\ \overset{\text{s.}}{7{\cdot}31} \text{ (as applied to E negative)}$$

$$\text{E, from Tables} = -\ \underline{15\ 17{\cdot}47}$$

$$\text{Corrected E} = -\ 15\ 10{\cdot}16 \text{ (as applied to apparent time)}$$

Hence

$$\text{L.A.T.} = \overset{\text{h.}}{20}\ \overset{\text{m.}}{54}\ \overset{\text{s.}}{10} + \overset{\text{m.}}{15}\ \overset{\text{s.}}{10{\cdot}16} = \overset{\text{h.}}{21}\ \overset{\text{m.}}{9}\ \overset{\text{s.}}{20{\cdot}16},$$

	h.	m.	s.	
∴ Astronomical reckoning	21	9	20·16	on the 15th
Deduct 12 h. . .	12	0	0	
L.A.T. (civil) . .	9	2	20·16	on the 16th.

(3) In longitude 60° west, what is the 'local mean time' corresponding to 3 h. 12 m. 10 s. 'local apparent time' on May 24th?

	h.	m.	s.	
Here L.A.T. on May 24th is	3	12	10	
Correction for longitude W	+ 4	0	0	
∴ Greenwich apparent time =	7	12	10	May 24th

The value of E. is required for this corresponding Greenwich instant, therefore from N.A. under G.A.N. May 24th, we find

$$\text{E} = \pm\ \overset{\text{m.}}{..}\ \overset{\text{s.}}{..}$$

$$\text{Hourly variation} = \pm\ 0\ ..$$

$$\therefore\quad \text{E at Greenwich instant} = -\ \overset{\text{m.}}{..}\ \overset{\text{s.}}{..} - \overset{\text{m.}}{0}\ \overset{\text{s.}}{..} \times \overset{\text{h.}}{7{\cdot}21} = -\ \overset{\text{m.}}{..}\ \overset{\text{s.}}{..}$$

Now deducting this value of E from the given L.A.T. we derive the corresponding 'local mean time' required.

Sidereal Time at Mean Noon. The first step, in converting local sidereal time to local mean time, is to obtain local sidereal time at local mean noon, that is to say at the instant of 0 h. 0 m. 0 s. of local mean time at the place of observation. If the place be in west longitude, the corresponding Greenwich mean time will be later. Suppose that the place of observation is 15°, or 1 h. west. Then the Greenwich *mean time* corresponding to noon at place of observation is 1 h. 0 m. 0 s. But the sidereal clock is continually gaining on the mean-time clock, at the rate of 9·8565 seconds of *sidereal* time per hour, so that to find the sidereal time of mean noon (S.T. of M.N.) at the place of observation we must *augment* the sidereal time of mean noon at *Greenwich*, at the rate of 9·8565 seconds per hour of *west* longitude and in proportion for minutes and seconds of longitude, and the sum will be the sidereal time at mean noon at the place of observation.

Conversely, if the place of observation is in *east* longitude we must *diminish* the sidereal time at mean noon at the *same* nominal date as the local date at the rate of 9·8565 seconds per hour of longitude.

G.S.T.	Greenwich sidereal time.	G.M.N.	Greenwich mean noon.
L.S.T.	Local sidereal time.	L.M.N.	Local mean noon.
S.N.	Sidereal noon.	N.A.	Nautical Almanac.

Conversion of Sidereal into Mean Time.

The conversion of 'sidereal' into 'mean time' is effected, then, as follows.

Rule—From the 'sidereal time' at the place of observation (adding twenty-four hours if necessary) *deduct the 'sidereal time' of 'local mean noon' and convert the remainder* (which is the 'sidereal time interval,' elapsed since 'local mean noon') *into its 'mean-time equivalent,' the result being the corresponding 'local mean time.'*

The above conversion is effected by the use of tables of equivalents, to be found at the end of the Nautical Almanac or in Chambers' Mathematical Tables, p. 433, where tables of 'acceleration' or 'retardation' of one kind of time or the other are given.

The following example shows the calculations necessary.

On the 1st May, 1898, in long 15° E. (one hour E.) the 'local sidereal time' is 20 h. 41 m. 12·65 s. Required the corresponding local 'mean time.'

		h. m. s.	h. m. s.
Sidereal local time above given is			20 41 12·65
G.S.T. of G.M.N. (from N.A.) . .		2 37 44·58	
Correction for acceleration for 1 hour = 15° E. long. at 9·86 s. per hour, (+ for W. and − for E.) . .	=	− 9·86	
L.S.T. of L.M.N.	=	2 37 34·72	
∴ Deducting			2 37 34·72
Gives 'sidereal interval' from L.M.N. .			18 3 37·93

∴ Taking out the M.T. equivalents from N.A. we have

		h. m. s.
18 h.	=	17 57 3·07
3 m.	=	0 2 59·51
37 s.	=	0 0 36·90
·93 s.	=	0 0 ·93
We have 'M.T. interval' from L.M.N. = L.M.T.	=	18 0 40·41

Conversion of Mean into Sidereal Time.

The conversion of 'mean' into 'sidereal time' is the converse of the above operation, and is effected as follows:—

From the given 'mean time' at the place of observation deduct the 'mean time' of *preceding* 'local sidereal noon,' and convert the *remainder*, which is the 'mean time interval' elapsed since preceding 'local sidereal noon,' into its 'sidereal time equivalent,' the result being the corresponding 'local sidereal time.' The 'equivalents' are taken out as above indicated.

The following example shows the calculations necessary.

On the 20th November, 1879, in longitude 165 W. (11 h.) the 'local mean time' is 10 h. 10 m. 18·85 s. Required the corresponding 'sidereal time.'

	h.	m.	s.	h.	m.	s.
Local mean time (astronomical reckoning) above given is				10	10	18·85
Greenwich mean time of *preceding* S.N. =	8	2	9·67			
Correction for 11 h. = 165° W. long. at 9·86 s. per hour (− for W. + for E.)	− 0	1	48·12			
L.M.T. of S.N. .	− 8	0	21·55			
Deducting				8	0	21·55
				2	9	57·30
By Chambers' Tables, correction for 2 h. =			19·71			
9 m. =			1·48			
57 s. =			·16			
			+ 21·35			+ 21·35
Local sidereal time required				2	10	18·65

In the above calculations it must be observed that the G.M.T. of 'sidereal noon' is found in the N.A. on page III. of each month, under the heading 'Transit of the 1st point of Aries.'

Care must be taken to take out the time of transit *preceding* the Greenwich instant corresponding to the time of observation.

Again, on or about the vernal equinoxes there may be two 'transits of the 1st point of Aries' in the same day, and when this occurs both times are entered in the N.A.

For full list of abbreviations, *vide* Appendix A.

CHAPTER IX.

DETERMINATION OF LATITUDE, TIME, AND AZIMUTH.

Corrections Applicable to all Observed Angles of Altitude.

Before the observed angles of the altitudes of heavenly bodies can be used for computation, the following corrections must first be applied in the order given.

(1) For index error, if a sextant be used. For level reading (E and O), if a theodolite be used.

(2) For instrumental and personal errors when known.

(3) Reduction of double altitudes (2-alts.), when a sextant with an artificial horizon is used.

(4) For refraction, in all cases (*vide* Appendix E).

(5) For semi-diameter, when the sun is observed.

(6) For parallax in altitude, when the sun is observed.

In making the above corrections, the latest corrected angle should be taken in each case as 'argument' for the next correction.

Corrections (1) and (3) have been already described.

Correction (2) depends on imperfections in the make of the instrument, as well as peculiarities in manipulation by the observer, quantities which can only be arrived at after continued practice with the same instrument.

Correction (5) for semi-diameter, is necessary in order to reduce observations taken to the sun's limb, to what they would be if its centre were observed to. The varying values of the sun's semi-diameter are given in the N.A., but can better be found when a sextant is used by actual measurement, at the time of taking a set of observations.

Correction (6) for parallax, is taken from a table, such as that given in No. IV. of Appendix E. It can also be calculated from the 'horizontal parallax' table, given in the N.A. page I., by multiplying the same by the nat. sine of the zenith distance.

Star Charts.

A set of star charts is given in Appendix C, in Plates numbered I. to IV.

In order to use the stars for observations, we must have some means of distinguishing them by the eye, especially when using a sextant. Knowing the R.A. and δ of any star, this is easily done by plotting its position on a *celestial map* or on a globe, in exactly the same way as a place is plotted on a

map or globe, only that 'R.A. circles' and 'declinations' are made use of, instead of 'longitude circles' and 'latitudes.'

From these maps or globes the *relative* positions of the various stars are seen at once, and then, their positions in the heavens can be found by referring these positions to imaginary lines joining well-defined stars or groups of stars, and following these lines in the heavens. With a little practice, any particular star can be found in this way with the aid of a star chart. The seven large stars of the constellation 'Ursa Major,' the 'pole star,' and some other groups, form excellent guiding marks by which to determine the positions of the other stars. Distances may be very accurately measured by the eye, when once accustomed to estimate rightly any known distance—as, for instance, the space between the two pointers of Ursa Major, which is about 5°.

In using a star chart we must be careful to consider under what conditions it is constructed.

The observer may be supposed to be standing (as he is in reality) at the centre of the celestial sphere, in which case the chart represents the stars as seen when looking upwards. When the chart, therefore, is placed on the table the stars will appear in their proper relative positions, but the E. and W. points will be reversed with regard to the *true* E. and W.

On the other hand, the observer may be supposed to be standing at a point outside the sphere, vertically over his supposed position on the earth, and looking down on it as in looking down on a globe. In this case the E. and W. points on the chart will correspond with the true E. and W. points, but the relative positions of the stars will be reversed, one to another, and they will not appear as they actually do in the heavens.

When reading a star chart it is always safe to consider the position of a star by its R.A. and declination shown on the chart. For instance, you can at once see which star lies to the E. of another by its R.A. (*i.e.*, S.T. of transit), as the *later hour* star must be the more *easterly* one, for it crosses the meridian *later*.

Observation of Meridian Altitude.

The '*modus operandi*,' when observing meridian altitudes with a theodolite, is as follows.

Direct the telescope of the theodolite to the star, whilst it is still a little to the east of the meridian, and intersect it with the cross-hairs, following it both in altitude and azimuth by turning the tangent-screws till it ceases to rise. Now, stop all movement in altitude, but continue to follow in azimuth until the star has distinctly commenced to dip. The reading of the vertical limb (the mean of both verniers), corrected for level error, is the meridian altitude of the star. This observed angle, corrected for refraction, is the true meridian altitude of the star.

Deducting the altitude from 90°, the zenith distance is obtained.

Computing Latitude.

The latitude is then computed by one of the following precepts.

(*a*) The latitude and declination being of the *same* name (*both* south or *both* north), and the observation being taken on the side of the zenith *away from the*

pole, *i.e.* to the south of the zenith in the northern hemisphere, to the north in the southern, then, latitude = zenith distance + declination.

(*b*) The latitude and declination being of *opposite* names (*one* south and the *other* north, or conversely), and the star being observed on the side of the zenith *away from the pole*, then, latitude = zenith distance − declination.

(*c*) The star being observed between the zenith and the pole, in other words at its *upper culmination*, then, latitude = declination − zenith distance.

(*d*) The star being observed *below the pole*, a lower culmination of a circum-polar star, then, latitude = 180° − (zenith distance + declination).

Example (*a*).

STAR SOUTH OF ZENITH.

Latitude north, declination north.

			°	′	″
			90	00	00
Deduct	Star's true altitude		66	54	10
	Remainder, star's zenith distance. .		23	05	50
Add	Star's declination		28	22	48
	Sum = latitude	=	51	28	38

(*b*) At the same place north latitude, star's declination S, observed south of the zenith.

			°	′	′
	Star's true altitude		12	25	33
Hence	Star's true zenith distance (deducting from 90°)	+	77	34	27
	Star's declination S	−	26	05	49
	Difference = latitude . . .	=	51	28	38

(*c*) At same place, star's declination N, star observed between zenith and pole.

			°	′	″
	Star's true altitude		78	55	10
Hence	Star's zenith distance (deducting from 90°)	−	11	04	50
	Star's declination.	+	62	33	27
	Difference = latitude . . .	=	51	28	37

(*d*) At same place, star below pole—

			°	′	″
	Star's true altitude		36	14	15
Hence	Star's zenith distance	+	53	45	45
	Star's declination	+	74	45	38
	Sum		128	31	23
	Deduct from 180°		180	0	0
	Remainder, latitude		51	28	37

General Proof.

In the attached figure 56, let the plane of the paper represent the plane of the meridian, O P being the polar axis, E O the plane of the equator, and H O N the plane of the horizon.

It is evident that the angle P O N is equal to the angle E O Z, the latitude of the place.

Let

h = the altitude of a star S = S O H
ζ = the zenith distance = Z O S
δ = declination of S } both + when north of equator,
ϕ = latitude of place } both − when south of equator.

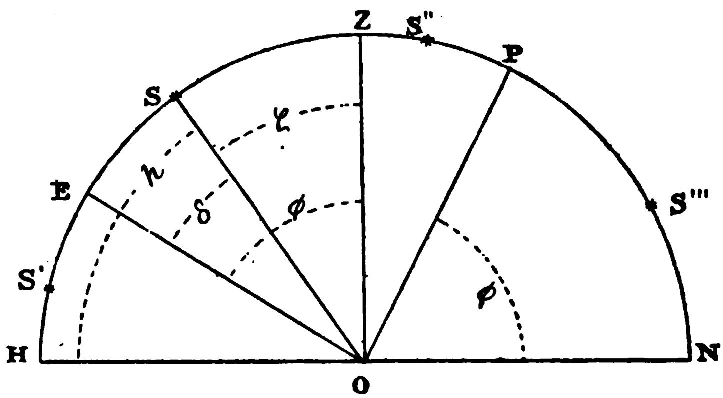


FIG. 56.

Now supposing other positions of the star, viz. S′, S″, S‴, with corresponding declinations δ', δ'', δ'''

We have the four cases—

ϕ calculated with star	S	$= (90 - h) + \delta$	$= +\zeta + \delta$	(showing proper signs)	
" "	S′	$= (90 - h) - \delta$	$= +\zeta + (-\delta)$	"	"
" "	S″	$= \delta - (90 - h)$	$= +\delta + (-\zeta)$	"	"
" "	S‴	$= \delta^* - (90 - h)$	$= +\delta^* + (-\zeta)$	"	"

Where δ^* = S‴ O E measured through the zenith and elevated pole
= the supplement of the declination given in the N.A.

It is thus seen that the general equation $\phi = \zeta + \delta$ holds good for all positions of a heavenly body S, provided the proper values and signs are given to ζ and δ.

Example of Latitude by Method of Single Meridian Altitudes.

We have, from the above, an easy approximate method of determining the latitude of a place, since we can find the declination of a star (used as a general term, including sun, moon, or planets), from the N.A., and obtain the zenith distance by measuring its meridian altitude.

Example.—On the 19th March, 1901, in longitude 72° 42′ E., the double altitude of the sun's lower limb was 89° 56′ 50″ (sun N. of zenith), sextant index

correction − 3′ 4″, bar. 29″·5. Thermometer 76° F. Find the latitude of the place.

	°	′	″
Obsd. 2-alt. . .	89	56	50
I. corr. . . −		3	4
2)	89	53	46
	44	56	53
*Refractn. −			54
	44	55	59
Semi-diam. +		16	5
	45	12	4
Parallax in alt. . . +			6
	45	12	10
	90	0	0
Zenith dist., or z. d.	44	47	50
δ at L.A.N. or sun's culmination		23	49·9
Lat. S (−)	45	11	39·9

	h.	m.	s.	
L.A.N. of obsn. .	0	0	0	
Long. E. . . . −	4	50	8	corresponding Greenwich instant on the 19th March.
G.A.T. of obsn. .	19	9	52	
=	19	15	0	
Mean refractn. . . .	0	58	00	
Corr. for bar. . −	0	0	97	say 4″
,, temp. . −	0	3	04	
Corrected refractn.	0	54	00*	(Appendix E).

	h.	m.	s.	
δ on 19th S	0	42	43·9	at G.A.N.
Difference for 19 h.·15 at 59″·29 decreasing	−0	18	54	
*δ at L.A.N. . . .	0	23	49·9	S (−)

* In general formula—

$$\pm \phi = (\pm) z + (\pm) \delta$$

N.B.—Here z. d. is minus, and δ is minus,

hence $$-\phi = -z - \delta.$$

If the star be near to the pole (Polaris, for example), its motion in altitude is so slow, when near to its meridian passage, that a pair of observations, face-right and face-left, may be made. When, however, the star is remote from the pole only a single altitude can be taken. A second observation of the same star may be made next night with face reversed, or another star, preferably at or about the same meridian altitude, may be similarly observed with face reversed. The mean latitude will be thus freed from collimation error.

Latitudes from Two Stars.

A still better method is to observe two stars, one culminating on the *north* and the other on the *south* side of the zenith, preferably at or about the same altitude. In this case the face should not be changed.

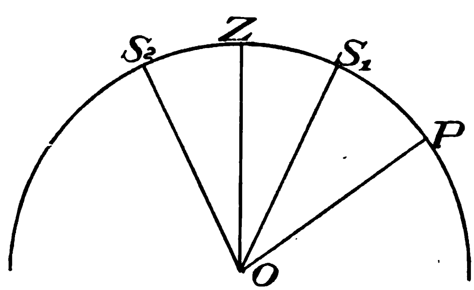


FIG. 57.

That is to say, having observed the first star, turn the theodolite round through 180° in azimuth to observe the second.

Let S_1 (fig. 57) be a star culminating between the zenith and the pole, and S_2 one culminating on the side of the zenith, remote from the pole. Let their respective zenith distances be z_1 and z_2, and their declinations d_1 and d_2.

Let the declination of S_2 be of the same name as the latitude. Then,

$$L = z_2 + d_2$$
$$L_1 = d_1 - z_1$$

Taking the mean of these determinations,

$$L_0 = \frac{d_1 + d_2}{2} + \frac{z_2 - z_1}{2} \quad . \quad . \quad . \quad . \quad (a)$$

Next, let the declination of S_2 be of opposite name to the latitude, and S_1 as before, culminating between the zenith and the pole. Then,

$$L = z_2 - d_2$$
$$L_1 = d_1 - z_1$$
$$L_0 = \frac{d_1 - d_2}{2} + \frac{z_2 - z_1}{2}$$

Again, let the declination of the star S_2 be of the same name as the latitude, and let S_1 be above the pole. Then,

$$L = z_2 + d_2$$
$$L_1 = 180 - (z_1 + d_1)$$
$$L + L_1 = 180 + z_2 + d_2 - (z_1 + d_1)$$
$$L_0 = 90 + \frac{d_2 - d_1}{2} + \frac{z_2 - z_1}{2}$$

Lastly, when S_2 and latitude have opposite names,

$$L = z_2 - d_2$$
$$L_1 = 180 - (z_1 + d_1)$$
$$L_0 = 90 - \frac{d_1 + d_2}{2} + \frac{z_2 - z_1}{2}$$

The result is, that in all cases, the quantity which is dependent on accuracy of measurement, is the *difference* of the zenith distances, not the zenith distances themselves. Consequently, if the two zenith distances be affected with the same error, say an error of collimation, making both too great or both too small, by the same amount, the difference, and consequently the latitude, will be correct. For this reason it is recommended that both northern and southern transits should be observed with the same face. By selecting several pairs of north and south stars, culminating at nearly equal altitudes, errors due to dislevelment of the telescope-axis of the instrument, will also be greatly eliminated.

To determine the latitude correctly, the procedure is to select from the

Nautical Almanac several *pairs* of stars, each pair culminating at or near the same zenith distance, one north and the other south of the zenith. The 'meridian altitudes' of each pair, should then be observed, with the same face. The mean of the whole set will be a very close approximation to the true latitude.

Further, it is by means of this method, that latitudes are determined with the greatest possible accuracy. The pairs of stars are so selected as regards declination, that stars of each pair have zenith distances so nearly equal and *so small* that, on rotating the theodolite through 180°, both stars appear successively within the field of the telescope, without the necessity for altering the angle of altitude. A micrometer is attached to the eye-piece of the telescope. By this method the difference of altitude (or what amounts to the same thing, the difference of zenith distance) is measured with great precision. In this way, with an instrument of comparatively moderate size, and a series of observations, latitude may be determined to a second or less, or to less than 100 feet. The instrument specially constructed for such observations is called a 'zenith sector.'

Latitude by Circum-meridian Altitudes.

When a number of altitudes are observed very near the meridian they are called *circum-meridian* altitudes.

Each altitude reduced to the meridian gives nearly as accurate a result as if the observation was taken on the meridian.

It can be shown by investigating the formulæ that near the meridian, the altitude varies as the *square* of the hour angle, and not simply in proportion to the time. Hence it is that near the meridian we cannot reduce a number of altitudes by taking their mean to correspond to the mean of the times, as is done (in most cases without sensible error), when the observations are remote from the meridian. We are obliged, therefore, to reduce *each* altitude separately and take the mean of all the results. How far off the meridian or zenith may observations be taken under this method? This question is fully investigated in Vol. I., "Chauvenet" (*see* also his Table VII. of limiting hour angles). But the following are good general rules.

General Rules.

(*a*) The zenith distance should never be less than 10°, and the hour angles should never exceed 20 minutes of time.

(*b*) The method cannot be used if δ and ϕ are almost equal, as might easily occur in the tropics with the sun.

(*c*) The closer to the meridian (and therefore the more rapidly the observations are taken) the better will be the results.

Instructions for Observing.

The observer should commence observing altitudes about 15 minutes before the transit, reversing the telescope after the *first* observation and then after *each pair* of observations.

In the case of the sun, altitudes are taken to the upper and lower limbs alternately.

The observations should be continued till as many altitudes have been obtained on the W. side of the transit as on the E. side.

The most favourable series of observations can afterwards be selected to compute from, as no greater accuracy can practically be obtained by computing from a large number of observations in preference to six or eight well situated with regard to the meridian.

For abbreviations, *vide* Appendix A.

It is not necessary to have an *equal* number of observations on either side of the meridian, but the better they are *balanced* the better will be the results. For example, three observations on one side of the transit and five on the other may be well balanced.

As the altitudes are changing very slowly contacts have to be *made* by the slow-motion screw.

Time of Transit Computed. The *time of transit* must be *computed* in order to obtain the hour angles for each observation.

In the case of a star, the R.A. (given in N.A.) is the *S.T. of transit.*

In the case of the sun, the L.S.T. of transit can be got from the Greenwich R.A. (as given in N.A.) when the latter is corrected for longitude and the L.M.T. of transit can be got from the *equation of time*, also corrected for longitude.

Chronometer Correction. The chronometer correction must be known, to give correct local time at the 'mean of the observed times,' and is usually obtained by an independent set of observations.

The chronometer correction can be found sufficiently accurately from the observations themselves, as follows.

Select two observed altitudes at times T and T', at as great a distance apart as possible. Putting $\sin^2 \frac{t}{2}$ (in the formula on page 152) $= \left(\frac{t}{2}\right) \times \sin^2 1''$, and multiplying by 15^2 to reduce time to arc, we have

$$z - z_1 = \frac{\cos \phi \cos \delta}{\sin z_1} \left(\frac{225 \sin 1''}{2}\right) t^2$$

$$= \mathrm{A}\, a\, t^2 \text{ when } a = \frac{225 \sin 1''}{2} \text{ and } \log a = 6{\cdot}736727$$

or $\bar{4}{\cdot}736727$.

hence

$$z_1 = z - \mathrm{A}\, a\, t^2$$

again

$$z_1 = z' - \mathrm{A}\, a\, (t')^2$$

or

$$(z' - z) = \mathrm{A}\, a\, (t'^2 - t^2) = \mathrm{A}\, a\, (t' - t)\, (t' + t)$$

Now, either $t' - t$ or $t' + t$ are evidently known, as well as $z' - z$, hence t' or t can be found, and therefore the correction at either observation.

Example.—From circum-merid. alts. of Procyon, 4/3/78, $\phi = 51° 23' 24''$.

	° ′ ″		h. m. s.
Observed 2-alts. on same side of meridian arc	88 8 20 = z		T = 7 18 8
	88 16 30 = z'		T′ = 7 24 11
	2) 8 10		6 3
			60
	$z' - z$ = 4 5 = 245″		$t' - t$ = 363ˢ

$$
\begin{array}{lll}
\text{Now} & \log A & = 9\cdot937326 \\
 & \log a & = \bar{4}\cdot736727 \\
 & \log (t' - t) & = \underline{2\cdot559907} \\
 & & \ 9\cdot233960 \\
 & \text{co-log} & = 0\cdot766040 \\
 & \log (s' - s) & \ \underline{2\cdot389166} \\
\end{array}
$$

$$3\cdot155206 = \log 1429\cdot6, \text{ or } t' + t$$
$$363\cdot0 \text{ is } t' - t$$
$$2\overline{)1066\cdot6} = 2t$$

		h.	m.	s.
$533\cdot3$ =	$t =$	0	8	53·3
Observed time	$T' =$	7	24	11·0
∴ Chron. time of transit of star =		7	33	4·3
Star's R.A. =		7	33	0·4
Chron. corr.	−			3s·9

The chronometer correction has to be applied to the observed *time* of transit to obtain the '*chronometer* time of transit.'

Rate of Chronometer.

The *rate* of chronometer is practically only required when the chronometer or watch used is unreliable, or when the *interval* between the 'time of transit' and the 'time at which the chronometer error was determined' exceeds one or two hours.

Example.—Error of watch determined by set of observations for time at 20 h. 5 m. 0 s. was found to be 2 m. 20 s. slow (i.e. correction + 2 m. 20 s.).

Rate of watch ascertained to average about 1·3 s. per hour, *gaining.*

'*Mean* of times of transit' 24 h. 10 m. 0 s. on the same day.

We have therefore,

Watch gained in elapsed interval 4·1 h. × 1·3 s. = 5 s.·33.

∴ Watch *error at time of transit* = 2 m. 20 s. less 5s·33.
= 2 m. 14 s. ·67 *slow.*

In observations of a star, the chronometer error is generally obtained by an independent set of observations, taken within one hour or less of the circum-meridian altitude observations, and any good ordinary watch may be relied on to keep sufficiently accurate time for such a small interval without any correction for *rate*, but it is advisable to ascertain the rate in case it should be unusually high. No good ordinary lever watch should gain or lose more than 12 seconds per diem.

Formula for Computation by Circum-meridian Altitudes.

The formulæ employed in the computation of latitude by circum-meridian altitudes are as follows.

Starting with the general formula for spherical trigonometry, giving the relations between one angle and three sides,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

For abbreviations, *vide* Appendix A.

Here we have (*vide* fig. 39, page 84)

$$A = ZPS = t = \text{hour angle}$$
$$a = \quad ZS = z = \text{zenith distance}$$
$$b = \quad PS = 90^\circ - \delta = \text{polar distance}$$
$$c = \quad PZ = 90^\circ - \phi = \text{co-latitude}$$

Then

$$\cos z = \cos(90^\circ - \phi)\cos(90^\circ - \delta) + \sin(90^\circ - \phi)\sin(90^\circ - \delta)\cos t = \sin\phi\sin\delta + \cos\phi\cos\delta\cos t$$

But

$$\cos t = 1 - 2\sin^2\frac{t}{2}$$

$\therefore$

$$\cos z = \sin\phi\sin\delta + \cos\phi\cos\delta - 2\cos\phi\cos\delta\sin^2\frac{t}{2}$$

And since

$$\sin\phi\sin\delta + \cos\phi\cos\delta = \cos\pm(\phi - \delta)$$

we have

$$\cos z = \cos\pm(\phi - \delta) - 2\cos\phi\sin\delta\sin^2\frac{t}{2}$$

As this reduction is simplified in the case of a star, since its declination remains practically constant, it will be considered first.

Let z° equal the meridian zenith distance.

Then

$$z^\circ = \pm(\phi - \delta)$$

and

$$\cos z = \cos z^\circ - 2\cos\phi\cos\delta\sin^2\frac{t}{2} \quad . \quad . \quad . \quad (a)$$

$$\cdot \qquad 2\sin\frac{z^\circ + z}{2} \cdot \sin\frac{z - z^\circ}{2} = 2\cos\phi\cos\delta\sin^2\frac{t}{2}$$

Developing this into a series, and neglecting the third and higher powers of $z - z_0$ we have, putting $\sin\frac{z^\circ + z}{2} = \sin z^\circ$.

$$z^\circ = z - \frac{\cos\phi\cos\delta}{\sin z^\circ} \times \frac{2\sin^2\frac{t}{2}}{\sin 1''} \quad . \quad . \quad . \quad (b)$$

Now tbe greatest observed altitude gives us the nearest approximation we can

get to z°, so, calling this *least* zenith distance z_1 and the corresponding value of latitude ϕ_1 we have

$$z^\circ = z - \frac{\cos\phi_1 \cos\delta}{\sin z_1} \times \frac{2\sin^2\frac{t}{2}}{\sin 1''}$$

Putting $A = \dfrac{\cos\phi_1 \cos\delta}{\sin z_1}$ and $m = \dfrac{2\sin^2\frac{t}{2}}{\sin 1''}$

we get

$$z^\circ = z - A\,m$$

Now for any other observations with zenith distances, z' z'', etc., and hour angles t' t'', etc.

$$z^\circ = z' - A\,m'$$
$$z^\circ = z'' - A\,m''$$

and so on, and the mean of x such observations is

$$\frac{x\,z^\circ}{x} = \frac{z + z' + z'' \ldots}{x} - \frac{A\,(m + m' + m''\,m \ldots)}{x}$$

Also if z_0 be the arithmetical mean of the observed zenith distances, and m_0 of the values m, then

$$z^\circ = z_0 - A\,m_0$$

and the latitude is calculated with this value z° for a zenith distance by the rule $\phi = z + \delta$. (Page 146.)

N.B.—If a star be observed at its *lower* culmination

then

$$z^\circ = 180^\circ - (\phi + \delta)$$

or

$$z^\circ = z_0 + A\,m^0$$

For the more lengthy proof of the formula for calculating latitudes by circum-meridian altitudes of the sun, the student is referred to Chauvenet's work on 'Spherical and Practical Astronomy.'

Observations for Latitude, N. and S. of the Zenith.

In order to eliminate instrumental errors, and those in calculating the correction for refraction, observations of stars N. and S. of the zenith and nearly the same altitude, should always be made when great accuracy is aimed at. In northern latitudes Polaris is generally selected as the N. star.

Examples.

Two examples are here given of latitudes found by circum-meridian altitudes of stars taken at places in different latitudes and longitudes.

Example in Lat. by Circ. Merid. Alts. of a Star.

At A in long. 5 h. 5 m. 8 s. E., and approximate lat. 54° N., on the 1st September, 1901, 'Altair' (α Aquilæ) was observed with a sextant (ind. corr. − 24′ 50″) and M. T. chron. (0 m. 36 s. fast). Bar. 30·07 in. Therm. 60° F.

	2 Alts. ° ′ ″	Chron. Times. h. m. s.	T_0 = 9h. 6m. 8s·7. $T_0 \sim T$.	*m.*
	94 49 0	9 1 17	4 51 7	46·4
	49 40	9 2 54	3 14 7	20·6
	49 50	9 4 36	1 32 7	4·6
	94 50 0	9 7 0	0 52 3	1·4
	49 20	9 9 11	3 2 3	18·1
	48 40	9 12 0	5 51 3	67·2
	48 10	9 14 30	7 21 3	106·2
Sums	(344) 40	(51) 28	7)	264·5
Means	94 49 14·3	9 7 21·1	Mean	37·78

	° ′ ″
I. corr.	0 24 50·0
2)	94 24 24·3
Corr. alt.	47 12 12·1
Refraction −	58·8
	47 11 13·3
	90 0 0
Mean z. d. or ζ_0	42 48 46·7

	″
Refractn. due to alt.	53·81
Corr. for bar. +	·12
Corr. for therm. −	1·07
Corr. refractn.	52·86

From above

	° ′ ″
Greatest obsd. 2 alt.	94 50 0
I. corr.	− 0 24 50
2)	94 25 10
	47 12 35
Refraction	− 0 0 52·8
	47 11 42·2
	90 0 0
Approx. z. d. or ζ_1	42 48 17·8
Dec. (δ)	+ 8 36 47·2
Approx. Lat. ϕ_1	51 25 5

Computation for True Lat.—

	° ′ ″	
Log cos ϕ_1	51 25 5	= 9·7949425
Log cos δ	8 36 47	= 9·9950657
Log cosec ζ_1	42 48 17·8	= 10·1670077
∴ Log A		= 29·9578159
Log m_0 (37·78)		= 1·5772620
∴ Log A m_0		= 1·5350779

and A m_0 = 34·28″

	° ′ ″
but ζ_0 =	42 48 46·7
−	34·3
and z_0 or $\zeta_0 - \mathrm{A}\,m_0$ =	42 48 12·4
Dec. (δ) =	98 36 47·0
∴ True Lat. N.	+ 51 24 59·4

Example in Lat. by Circ. Merid. Alts. in Lat. S., and Long. W.

At X (in South America), long. 4 h. 20 m. 0 s. W., and approximate lat. 30° S., on the 10th November, 1901, circum-meridian alts. of the sun's upper limb were taken as follows with an 8″ sextant (ind. corr. — 45″), and a M.T. chron. with corr. − 1 m. 10 s. at noon. Bar. 30·75 in. Therm. 75° F.

2 Alts. ° ′ ″	Times. h. m. s.	$T_0 \sim T$.	m.
84 37 15	23 39 30	5 38 7	62·4
0 38 0	23 41 30	3 38 7	26·0
0 38 10	23 43 30	1 38 7	5·3
0 38 21	23 45 30	0 21 3	·2
0 38 11	23 47 30	2 21 3	10·8
0 37 55	23 49 30	4 21 3	37·3
0 37 35	23 51 30	6 21 3	79·3
Sums . 3(58) 27	(318) 30		221·3
Means 84 38 21	23 45 30		13·6 m_0

	h. m. s.
Lat. of L.A.N. . . .	0 0 0
E. of time	0 16 2·3
0·24 × 4·33 for W. long. + *decreasing*	+0 0 1·0
L.M.T. of L.A.N.* .	23 43 58·7
Chr. fast	+0 1 10
T_0	23 45 8·7

Mean of Hour Angles.

	h. m. s.
(*a*) Mean of time .	23 45 30
(*b*) Chr. T_0	23 45 8·7
(*c*) Mean of $T_0 \sim T$.	+ 0 21 3

(*c*) is + if (*a*) later than (*b*).

Time Intl. of Obsn. from G.M.N.

	h. m. s.	
* From above L.M.T. of L.A.N.	23 43 58·7	of 9th.
Corr. for long. W.	+ 4 30 0	
G.M.T. of L.A.N.	4 13 58·7	of 10th.
Mean of $T_0 \sim T$	+ 0 21 4	
Intl. of mean of obsn. from G.M.N.	4 14 20·0	
= in hours . .	4·24	

Declination.

	° ′ ″
	− 17 1 55·6
Corr. 42·51 × 4·24	+ 0 3 0·3
δ = . .	−17 4 55·9

Refn.	′ ″	*Par.*	″
due to alt.	1 3·9	hor. par.	7·0
corr. bar.	+ 1·0	×	
corr. ther.	− 3·0	cos *h*	·74
corrctd.	1 1·9	corrctd.	5·2

To compute approx. lat. ϕ_1

	° ′ ″
Max. obsd. 2 alt	84 38 21
I. corr.	− 0 0 45
	2) 84 37 36
Corr alt.	42 18 48·0
Refractn. ∽ diam. par. in alt. . . .	− 0 17 7·7
True approx. altitude	42 1 41·3
	90 0 0
App. z. d. or ζ_1	47 58 18·7
Dec. (δ)	−17 4 55·9
Approx. lat. ϕ_1	30 53 22·8

Mean z. d. or ζ_0

	° ′ ″
Mean 2 alt.	84 38 21
I. corr.	− 0 1 10
	2) 84 37 11
Alt.	42 18 35·5
Refrac.	− 0 1 1·9
Semi-diam. . . .	− 0 16 11·0
Parallax	+ 0 0 5·2
Mean alt.	42 1 27·8
	90 0 0
Mean z. d. or ζ_0 . .	47 58 32·2

log cos ϕ . .	30 53 23 . .	9·9335705
log cos δ . .	17 4 56 . .	9·9804127
log cosec ζ_1 . .	47 58 19 . .	9·9257443
log A	=	39·7397275
log m_0 (31·6) . . .		1·4926871
log A m_0		1·2394146

	° ′ ″
A m_0 . . .	= − 0 0 17·4
ζ_0	= 47 58 33·2
z_0	= 47 58 15·8
Dec. (δ) .	= −17 4 53·9
True lat. (ϕ) S.	30 53 19·9

Formula for Calculating Latitude by Altitudes of Pole Star.

Latitude can be found by observations of the pole star, when out of the meridian, and the adaptation of the equation in spherical trigonometry which is used, is here investigated. Correct local time (t), an approximate value of lat. (ϕ), and the altitude (h) must be known.

Substituting as in the formula for circum-meridian altitudes, we have

$$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t$$

Now, $90 - p = \delta$, or $\sin \delta = \cos p$, and p (the polar distance) being very small (about 1° 15′), we can develop ϕ in a series of ascending powers of p, using as many terms as necessary to obtain any desired degree of precision.

Now h the altitude observed cannot differ from ϕ more than the polar distance (p), or 1° 15′, so let x equal this difference, and

$$\phi = (h - x)$$

Now

$$\sin \phi = \sin (h - x)$$

(developing)

$$= \sin h - x \cos h - \frac{x^2}{2} \sin h, \text{ etc.}$$

Similarly,

$$\cos \phi = \cos h + x \sin h - \frac{x^2}{2} \cos h, \text{ etc.}$$

and

$$\sin p = p - \frac{p^3}{6}, \text{ etc.}$$

$$\cos p = 1 - \frac{p^2}{2}, \text{ etc.}$$

Substituting these values in the equation

$$\sin h = \sin \phi \cos p + \cos \phi \sin p \cos t,$$

we have

$$\begin{aligned} \sin h &= (\sin h - x \cos h, \text{ etc.}) \left(1 - \frac{p^2}{2}, \text{ etc.}\right) \\ &\quad + (\cos h + x \sin h, \text{ etc.}) \left(p - \frac{p^3}{6}, \text{ etc.}\right) \cos t \\ &= \sin h - x \cos h, \text{ etc.} + p \cos t \cos h \\ &\quad - \tfrac{1}{2} (p^2 - xp \cos t) \sin h, \text{ etc.} \qquad (a) \end{aligned}$$

And from this we get the general expression for the correction x, viz.

$$x = p \cos t - \tfrac{1}{2} (p^2 - xp \cos t) \tan h + \text{etc.}$$

For the first approximation, we take $x = p \cos t$, and substituting this value in equation (a) we get a second approximation $x = p \cos t - \frac{1}{2} p^2 \sin^2 t \tan h$. Now, by differentiating, it can be proved that the first two terms give a result within 1″ of the truth, hence for this degree of accuracy we have the formula

$$\phi = h - p \cos t + \tfrac{1}{2} p^2 \sin 1'' \sin^2 t \tan h \qquad (b)$$

In formula (a) x and p are in circular measure, but are reduced to seconds in (b) by the factor sin $1''$.

Calculation from Appendix F provides for the reduction of observations by this formula. See also Tables I., II. and III. at the end of the N.A., which are therein decribed.

Rough Method without knowledge of Declination of Star.

A useful method of finding latitude which does not entail a knowledge of the declination of any celestial body, is as follows :—

Latitude by Circum-Polar Star at time of Transit.

Select a circum-polar star and observe its altitude at both upper and lower transit, the mean of these altitudes (corrected for refraction, etc.) will give at once a value for the latitude of the place. Since, however, two such observations at upper and lower transit must necessarily be separated by a period of 12 hours, it is not always practicable to make them.

DETERMINATION OF TIME.

Relation between Time and Hour Angle.

We have seen that time, either solar or sidereal, is dependent on the hour angle of the sun or the 'first point of Aries.' Since we are able by means of astronomical observations of the sun, to compute its hour angle, solar time is readily found, and similarly, with a star, by means of its R.A. (which is given in the N.A.) the hour angle of the 'first point of Aries' can be readily obtained.

We have, however, no method of measuring the hour angle easily, and accurately, by direct measurement, except when the heavenly body is on the meridian, *i.e.*, when the hour angle is zero.

If we were able to define the meridian plane sufficiently accurately to show the exact moment the star crosses it, we should obtain the simplest determination of time open to us. In ordinary field work, this can seldom be done, so we are reduced to other methods of obtaining the hour angle.

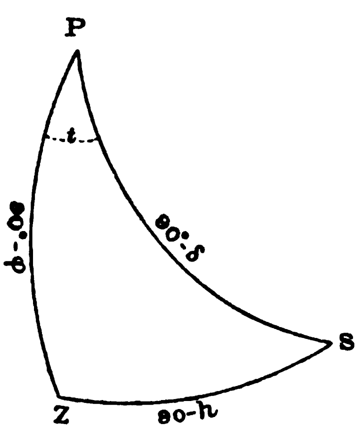


FIG. 58.

Observing Altitudes for Time.

An examination of the astronomical triangle P S Z (fig. 58), shows us that the value of the hour angle Z P S depends to a great extent upon the side Z S, for, of the other sides, P Z is constant, and P S varies only slightly with the change of declination, during the time taken in making an observation. Now Z S is the complement of the altitude, and thus in a position of the heavenly body, such as is shown in the figure, the hour angle changes in accordance with the changes of the altitude.

Obviously the most favourable period to observe altitudes for time will be

when they are changing as rapidly as possible, *i.e.*, when for a given change of altitude, the corresponding alteration in time, is as small as possible.

Now the altitude is changing fastest (and at the same rate for all stars observed from the same spot), when the star is on the prime vertical, hence, this is the best position in which to observe it.

Altitude of Stars on the Prime Vertical.

It is sometimes convenient to be able to compute the altitude of a star when it crosses the prime vertical. This can be done from the following formula,

$$\sin h = \frac{\sin \delta}{\sin \phi}$$

(*Vide* page 128.)

Limit of Accuracy in Time Determinations.

Time should be found by either of the methods hereafter described to an accuracy of less than one second of time, and may be applied for correction of the chronometer or whatever timekeeper the observer is using.

Chronometer Error.

Instead of altering the chronometer it is generally sufficient to record its *error* (or correction), which for any instant is the difference between the time it shows and the time it should show, this error being styled *fast* when it is ahead of the proper time, and *slow* when it is behind it. The error of a chronometer is generally recorded for some convenient *epoch*—such as mean noon—on a given day, *i.e.*, at the commencement of a given astronomical day. The 'correction' is that which should be *added to* or *deducted from* the time indicated.

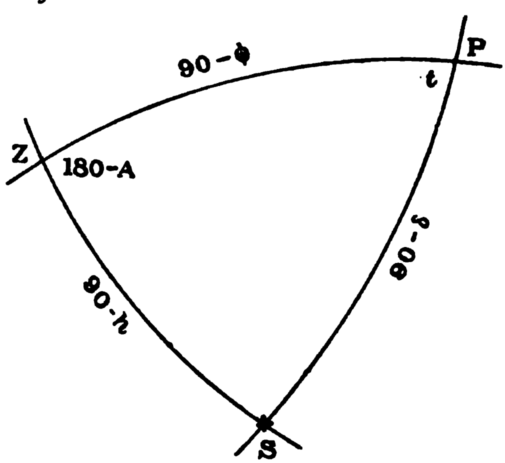


FIG. 59.

Time by Single Altitudes. Data Required.

In this case the data required to be known, in addition to the altitude observed, are, P Z the co-latitude (from best value of latitude known)—(*vide* fig. 59), and P S the polar distance found from the N.A. by assuming an approximate value for L.M.T. and longitude. It may be observed that a small error in the assumed latitude, and even a considerable error in the assumed longitude would produce very little error in the result.

Computation.

The following equation is used in the computation for time.

$$\tan \frac{t}{2} = \sqrt{\cos S \sin (S - h) \operatorname{cosec} (S - \phi) \sec (S - p)}$$

$$\text{where} \qquad S = \frac{h + \phi + p}{2}$$

(*Vide* page 85.)

N.B.—Latitude ϕ always positive, and p reckoned from the *elevated* pole.

General Instructions for Observing.

For precision several altitudes should be observed, and the *mean* of these, with the corresponding *mean* of the times, should be taken for computation. If the time assumed is much out, the computation must be repeated with new functions obtained by means of the corrected time.

The most favourable position for the observer for taking his observations is on the equator, the worst at either of the poles.

Stars which *cross* the prime vertical are most favourably placed for observing *when crossing* the prime vertical.

A star which changes its altitude less than five seconds of an arc in one second of time should not be used for obtaining time. Thus a star nearer than 20° to the pole should not be used for this purpose.

When the latitude is undetermined, a small error in its assumed value has no appreciable effect on the results of the observation if the stars are observed when near the prime vertical.

To eliminate instrumental errors, errors of refraction, and assumed functions, observations must be taken in pairs on either side of the meridian, in the case of the sun by morning and afternoon observations, in that of the stars by two stars, one *east* and one *west* of the meridian.

Instructions for Sextant Observations.

Take six or eight double altitudes at sufficiently short intervals to ensure that the altitudes change proportionately to the times. It is easy to do this without undue haste, by taking an observation every two minutes or so.

If observing the sun, take half the number of observations of the lower limb and half of the upper limb.

Instead of making contact with the slow motion screw, in fair weather, and when there are no clouds to interfere with the observation, set the vernier forward on the arc, an equal amount for each observation, and allow the heavenly body observed to make contact of itself.

Instructions for Theodolite Observations.

When observing the sun, adjust the instrument, turn the telescope on the sun, and make contact with the upper or lower limb on the centre or horizontal wire, noting the time of observation, and the readings of both ends of the level (O and E) and of the vertical circle, then reverse the telescope, and make contact with the other limb, noting readings as before.

This completes one set of observations, and can be repeated as often as required, for accuracy.

The procedure for observing a star is the same, except that the observations are taken by bisecting the star with the intersection of the two centre cross-wires.

Four observations complete a set, half on each face of instrument, and can be repeated as often as required.

Examples. Two examples are here given in finding the 'chronometer correction' by observations of stars (in the east and in the west), at places situated in different latitudes and longitudes (N. and S., E. and W., respectively).

Examples in Time Observations. I.—At X in South America (in long. 4 h. 20 m. W., and lat. 32° S.), the following observations of two stars for time were taken, on the 1st June 1901, with a sextant (i. corr. − 1′ 20″) and a mean time chronometer, which was supposed to be 3 minutes slow at noon (the epoch for the correction), and having a gaining rate of 0·6 sec. per diem.

(1) α Coronæ (rising).

Mean 2 alt. = 84° 9′ 48″. Mean of times = 10 h. 24 m. 19 s.
Bar. 29·6 in. Therm. 70°.

(2) β Centauri (setting).

Mean 2 alt. = 71° 29′ 48″. Mean of times = 12 h. 6 m. 4 s.

Elements for Calculations.

		′ ″		° ′ ″
For (1)	Refrac. for alt.	1 4·5	R.A. = 19 h. 2 m. 48 s.	
	Bar. corr.	− 0 1·0		
	Therm. corr.	− 0 3·0	Dec. (δ)	38 3 17·6
				90 0 0
	Corr. refrac.	1 0·5	Polar dist. (p)	51 56 42·4
For (2)	Refrac. for alt.	1 21·0	R.A. = 12 h. 3 m. 13 s.·5.	
	Bar. corr.	−0 1·0		
	Therm. corr.	−3 3·0	Dec. (δ)	50 10 16
				90 0 0
		1 17·0	Polar dist. (p)	39 49 44

		h. m. s.
G.S.T. of G.M.N.	=	4 37 5·7
Corr. for long. W. 9 s.·86 × 4 h.·33	+	42·7
L.S.T. of M.N.		4 37 48·4

Computations	(1) ° ′ ″		(2) ° ′ ″	
Mean obsd. alt.	42 4 54·0		35 39 54	
Corr. refracn.	− 0 1 0·5		− 0 1 17	
Corr. alt. (h)	42 3 53·5		35 38 37	
(ϕ)	32 0 0		32 0 0	
(p)	51 56 42·4		39 49 44	
	2) 126 0 36		2) 107 28 21	
Take log cos (s) =	63 0 18	9·6569744	53 44 10·5	9·7719585

	° ′ ″		° ″	
Take log cos (s) . . =	63 0 18	. . . 9·6569744	53 44 10·5	. . . 9·7719585
log sin $(s-h)$. =	20 56 24·5	. . . 9·5531425	18 5 33·5	. . . 9·4921342
log cosec $(s-\phi)$. =	31 0 18	. . . 10·2880977	21 44 10·5	. . . 10·4314085
log sec $(s-p)$. =	11 3 35·5	. . . 10·0080677	13 54 26·5	. . . 10·0129212
		2) 19·5062823		2) 19·7084624
	log tan $\frac{t}{2}$ =	9·7531911	log tan $\frac{t}{2}$ =	9·8542312

	h. m. s.
∴ $\frac{t}{2}$ in arc =	29° 31′ 56″
	2
5)	59 3 52
3)	11 40 40
H. A. at obsn.	3 56 15
Star's R. A. =	19 2 48
L. S. T. of obsn. . . .	15 6 33
L. S. T. local mean noon	4 37 48·4
S. T. interval . . . =	10 28 44·6

	h. m. s.
∴ $\frac{t}{2}$ in arc =	35° 33′ 36″
	2
5)	71 7 12
3)	14 13 26
H. A. at obsn.	4 44 28
Star's R. A. =	12 3 13
L. S. T. of obsn.	16 47 41
L. S. T. local mean noon	4 37 48·4
S. T. interval . . =	12 9 52·6

		h. m. s.
M. T. equivalents	hrs. .	9 58 21·7
	mins. .	0 27 55·4
	secs. .	0 0 43·9
∴ M. T. interval from noon =		10 27 1·0
Chron. time		10 24 19·0
∴ Chron. slow		0 2 42
		0 1 48·3
		2) 0 4 30·3
Mean corrctn. at time of obsn.		+ 0 2 15·15
To reduce to noon as the epoch for the corrctn.		− 0 0 0·25 gaining rate
Chron. corrctn. at noon		+ 0 2 14·90 (chron. slow)

		h. m. s.
M. T. equivalents	hrs.	11 58 2·0
	mins.	0 8 58·5
	secs.	0 0 51·8
∴ M. T. interval from noon		12 7 52·3
Chron. time		12 6 4
∴ Chron. slow		0 1 48·3

II.—At X in Siberia (in long. 5 h. 5 m. 8 s. E., and lat. 51° 25′ N.), the following observations of two stars for time, were taken on the 22nd February, 1901, with a theodolite and a mean time chronometer, which latter was supposed to be 3 minutes fast at noon (the epoch for correction), and had a gaining rate of 1·6 s. per diem.

(1) α Ceti (setting).

Mean of alts. = 22° 5′ 20″. Mean of obs. times = 8 h. 45 m. 10 s.
Bar. 29·6 in. Therm. 75°.

(2) β Leonis (rising).

Mean of alts. = 26° 30′ 10″. Mean of obs. times = 9 h. 7 m. 15 s.

Elements for Calculations.

		′	″
For (1) Refractn. for alt.		2	22·7
Bar. corrctn.	−	0	2·0
Therm. ditto	+	0	7·0
		2	27·7

R. A. = 2 h. 57 m. 7·3 s.

	°	′	″
Dec. (δ) =	3	42	2·6
	90	0	0
Polar dist. (p)	86	17	57·4

		′	″
For (2) Refractn. for alt.		1	56·4
Bar corrctn.	−	0	2·0
Therm. ditto	+	0	7·0
		2	1·4

R. A. = 11 h. 44 m. 3·1 s.

	°	′	″
Dec. (δ) =	15	7	13·2
	90	0	0
Polar dist. (p)	74	52	46·8

		h.	m.	s.
G. S. T. of M. N.		22	6	46·9
Corrctn. for Long. E. 9·86 s. × 5·08 h. (− E).	−	0	0	50·1
L. S. T. of M. N.		22	5	56·8

Computations	(1) ° ′ ″		(2) ° ′ ″	
Mean obsd. alt.	22 5 20·0		26 30 10·0	
Corrctn. for refractn.	− 0 2 27·7		− 0 2 1·4	
Corrected alt. (h)	22 2 52·3		(h) 26 28 8·6	
(ϕ)	51 25 0·0		(ϕ) 51 25 0·0	
(p)	86 17 57·4		(p) 74 52 46·8	
	2) 159 45 49·7		2) 152 45 55·4	
Take log cos (s)	79 52 54·8	9·2447148	76 22 57·7	9·37187237
log sin ($s-h$)	57 50 2·5	9·9276285	49 54 49·1	9·8837037
log cosec ($s-\phi$)	28 27 54·8	10·3212230	24 57 57·7	10·3746046
log sec ($s-p$)	6 25 2·6	10·0027298	1 30 10·9	10·0001494
		2) 19·4968961		2) 19·6303300
	log tan $\frac{t}{2}$ =	9·7481480	log tan $\frac{t}{2}$ =	9·8151650

∴ $\frac{t}{2}$ in arc = 29° 14′ 47″·6 $\qquad$ $\frac{t}{2}$ in arc = 33° 9′ 35″

(1)	(2)
29 14 47·6	33 9 35
2	2
5) 58 29 35·2	5) 66 19 10
3) 11 41 55·0	3) 13 15 50

		(1) h. m. s.		(2) h. m. s.
	H. A. of obsn.	3 53 58·3	H. A. of obsn.	4 25 16·6
but	Star's R. A.	= 2 57 4·3	but Star's R. A.	= 11 44 3·1
	L. S. T. of obsn.	6 51 2·6		7 18 46·5
	L. S. T. of L. M. N.	22 5 56·8		22 5 56·8
	S T. interval	8 45 5·8		9 12 49·7

		h. m. s.			h. m. s.
M. T. equivalents	8 hrs. =	7 58 41·36		9 hrs. =	8 51 31·53
	45 mins. =	0 44 52·63		12 mins. =	0 11 58·03
	5 secs. =	0 0 4·98		49 secs. =	0 0 48·36
	·80	0 0 0·80		·70 =	0 0 0·70
L. M. T. of obsn. . . .		8 43 39·77			9 4 18·62
Chron. time		8 46 10·00			9 7 15·00
Chron. fast		0 2 30·23	Chron. fast		0 2 56·38
		0 2 56·38			
		2)0 4 (8)6·61			
Mean corrctn. at time of obsn.		0 2 43·30			
Reduction to noon as epoch, at 1·5 s. per diem. gaining .		0 0 0·55 gaining rate			
Mean corrctn. at noon . .		−0 2 42·75 (chron. fast)			

Time by Equal Altitudes of a Star.

If a star (the declination of which is constant during the two observations) be observed before and after transit over the meridian when at the same altitude, it is obvious that its hour angle at both instants of observation must be equal to each other though on opposite sides of the meridian, and that the mean of these times of observation will give the actual time that the star was on the meridian.

As a star rises in altitude we may record an instant (say t), which by our timekeeper corresponds to the time that the star reaches a certain altitude, then, by waiting till the star has crossed the meridian and descends to the same altitude, and again recording the instant (say t'), we can find the moment of transit, which was $\frac{t+t'}{2}$ A comparison of this time (reduced to S.T.) with the R.A. of the star, determines for us the error of our timekeeper.

This method of equal altitudes is very accurate, and is independent of an accurate knowledge of latitude or declination, or of errors in collimation, or in the graduation of the instrument. It is (with a star) very simple in its procedure, but requires a clear sky for a considerable interval of time, and for altitudes less than about 45°, it is liable to variations on account of changes in temperature, and therefore, of 'refraction.'

Time by Equal Altitudes of the Sun.

When the sun is used, a correction has to be made for the change of declination that occurs between the times of observation, before and after the passage over the meridian.

Having obtained the 'mean of the two times' of observation, it may be corrected to the 'time of apparent noon,' *i.e.*, time of transit, by the following formula

$$\text{Mean of times} \pm \frac{t \times d\delta}{15} (\tan \phi \operatorname{cosec} t - \tan \delta \cot t)$$

where

t = half the interval between the two observations
$d\delta$ = the change of declination in one hour
δ = the declination of the sun when on the meridian.

This correction is *positive* when the sun is *leaving* the 'elevated pole,' in declination, *negative* when it is *approaching* the 'elevated pole.'

Instructions for Observing Equal Altitudes.

When taking observations of equal altitudes of stars, or the sun, instead of one observation only being made on each side of the meridian, it is necessary, for precision, to obtain a set as in observing for single altitudes. The instrument must be set forward and readings taken at intervals of 10, 20, or 30 minutes of arc, on the upward journey of the heavenly body, and to these readings the vernier can be easily adjusted on the downward journey of the sun or star. The 'mean times' of each set of observations will then be used as if they were the instants corresponding to one pair of altitudes only.*

Observations for Azimuth.

General Remarks.

To determine the direction of the meridian line, or the azimuth of any line, joining a given point with the station at which observations are being taken, it is usual to select some well defined object suitably situated, and at a moderate distance, to which all azimuth angles can be referred. Such an object is termed a 'referring object,' and is noted as such in the angle book, as R.O.

Azimuth by Single Altitudes.

Observations for the determination of azimuth by single altitudes of the sun or a star should be made when the heavenly body is near the prime vertical (as for time), for its horizontal motion or change in azimuth in a given time is then a 'minimum,' hence the effect of an error in time has the least effect on the result. A 'rising' and 'setting' star should be observed when great accuracy is desired. Observations can also be taken to a star *at* or *near* 'elongation,' or when less accuracy is aimed at, by observing the pole star at the instant of its upper or lower culmination.

Best Instrument for Azimuth Observations.

There is no doubt but that the transit theodolite is the instrument best adapted for azimuth observations, since it possesses both a 'horizontal' and a 'vertical' limb, and angular measurements in both a 'horizontal' and a 'vertical' plane are essential to the solution of these problems.

Azimuths can be determined with a sextant, by observing the inclined angle between an R.O. and the sun (these observations are very difficult to obtain with stars), having also measured the sun's angular altitude, as well as that of the R.O. Since by this method time must be accurately known, the watch correction must be determined just before, or after the observations.

Formulæ employed when Calculating Azimuths.

The following formulæ are employed in calculating azimuth when single altitudes of the sun or a star are observed.

$$(1) \quad \tan A = \frac{\tan t \cos M}{\sin(\phi - M)} \text{ and } \tan M = \frac{\tan \delta}{\cos t},$$

where A is measured from 0° (the depressed pole) to 360° westwards

* *Vide* Appendix H.

t is hour angle to twenty-four hours reckoned from upper transit

and

δ, ϕ, and t are known (*vide* Appendix A)

This formula is only used for azimuths by observations of the sun, and correct time must be obtained by independent observations.

$$(2)\quad \tan\frac{A}{2} = \sqrt{\sec s \sec(s-p)\sin(s-\phi)\sin(s-h)}$$

where $s = \dfrac{h+\phi+p}{2}$, and A is measured from the *elevated* pole.

This formula is used for azimuths by observations of the sun or of a star. (*Vide* Appendix E, Form 5.) (*Vide* Appendix A for abbreviations.)

When a circum-polar star is observed at elongation

(a) $\sin h = \sin\phi \sec p$ (useful to find the star)
(b) $\cos t = \tan\phi \tan p$ (to arrange time of observation)
(c) $\sin A = \sec\phi \sin p$.

The following examples show the working of these calculations.

Examples in Determination of Azimuth.

I. At Brisbane (Australia), in lat. 27° 28′ 15″ S., and long. 10 h. 11 m. 20 s. E., the following observations were taken on the 21st August, 1901, to determine with a theodolite the azimuth of a selected R.O. from a given station.

	°	′	″
(1) Star θ Centauri (setting in W.)			
Mean of alts. corrected for level reading (E and O)	39	10	0
Bar. 29·00″. Therm. 82° F.			
Hor. angle from R.O. to mean of observations (measured to the right)	101	10	0
(2) Star α Piscis Australis (rising in E.)			
Mean alt. as above	38	20	0
Bar. and therm. as above.			
Hor. angle from R.O. to mean of observations (measured to the left).	35	15	0

Elements for Computation.

	(1)	(2)
	′ ″	′ ″
Refractn. due to alt.	1 11·5	1 13·6
Corr. for bar.	−0 2·0	−0 2·0
Corr. for therm.	−0 4·0	−0 4·0
	1 5·5	1 7·6
	° ′ ″	° ′ ″
Declin. on 21st August, 1901 .	35 53 16·6	30 8 21·1
	90 0 0·0	90 0 0·0
Polar dist. (p)	54 6 43·4	(p) . 59 51 38·9

Computations.

	(1) ° ′ ″		(2) ° ′ ″
Mean obsd. alt.	39 10 0·0		38 20 0·0
Correctn. for refractn. .	− 0 1 5·5		− 0 1 7·6
True alt. (h)	39 8 54·5	(h)	38 18 52·4
Lat. (ϕ)	27 28 15·0	(ϕ)	27 28 15·0
Polar dist. (p)	54 6 43·4	(p)	59 51 38·9
	2) 120 43 52·9		2) 125 38 46·3
(s)	60 21 56·5		62 49 23·1
($s - p$)	6 15 13·1		2 57 44·2
($s - \phi$)	32 53 41·5		35 21 8·0
($s - h$)	21 13 2·0		24 30 30·7
log sec	10·3058667		10·3403310
log sec	10·0025928		10·0005808
log sin	9·7348789		9·7623793
log sin	9·5585951		9·6178682
	2) 19·6019335		2) 19·7211593
log tan $\frac{A}{2}$ =	9·8009667		= 9·8605796
and $\frac{A}{2}$ =	32 18 27·9	$\frac{A}{2}$ =	35 56 26·1
	2		2
(Setting) A . . =	64 36 55·8	(Rising) A =	71 54 52·2
Angle from R.O. =	101 10 0·0		35 15 0·0
∵ Azimuth of R.O. from elevated pole =	36 33 4·2		36 39 52·2
			36 33 4·2
			2) 72 (7)2 56·4
		Mean azimuth of R.O. . . . =	36 36 28·2

II. At Denver, in North America (in lat. 39·30 N., and long. 6h. 58m. 0s. W.) the following observations of the sun were taken on the 26th May, 1901, with a theodolite, in the afternoon, to determine the azimuth of a selected R. O. Bar. 30·2 inches, therm. 40°, mean altitude of sun's centre 38° 5′ 0″. Mean time of observations 7 h. 51 m. 15 s., by an S.T. chron., 2s·5 fast at noon, and with a gaining rate of 0″·8 per diem.

Elements for Computation.

	′ ″
Refraction due to alt.	1 14·3
Bar. corr.	+ 0 1·0
Therm. corr.	+ 0 1·5
	1 16·8

	h.	m.	s.
G.S.T. of G.M.N.	4	13	25·42
Corr. for W. long. + 9·86 s. per hour (6·96 h.) . .	+0	1	8·62
L.S.T. of L.M.N.	4	14	35·04
S.T. of obsn.	7	51	12·40
M.T. of obsn.	3	36	37·36
Long. W.	6	58	0·00
G.M.T. of obsn.	10	34	37·36

∴ Intl. from G.M.N. = 10·57 h.

	h.	m.	s.
Mean time of obsns. from field-book	7	51	15
2·5 s. fast + 0·1 s. gaining to time of obsn.	−	0	2·6
	7	51	12·4

	°	′	″
Dec. at G.M.N. (+N. −S)	+ 21	3	13
Hourly variation, 26″·41 × 10·57 h.	+	0	4 31
Dec. at time of obsn. . . .	21	3	44
	90	0	0
Polar dist. (p)	68	56	16

Computation.

	°	′	″
Mean obsd. alt.	38	5	0
Corr. for refractn. . . .	0	1	16·8
True alt. (h)	38	3	42·2
Latitude (ϕ)	39	30	0·0
Polar dist. (p)	68	56	16·0
2)	146	29	58·2
(s)	73	14	59·1
($s-p$)	4	18	43·1
($s-\phi$)	33	44	59·1
($s-h$)	35	11	16·9

log sec	10·5402145
log sec	10·0012310
log sin	9·7745464
log sin	9·7605742
2)	20·0765661
log tan $\frac{A}{2}$	10·0382830

	°	′	″
∴ $\frac{A}{2}$ =	47	31	19
			2
∴ A =	95	2	38
Angle from R.O. to mean obsn. from field-book	+1	2	10
Azimuth of R.O.	96	4	48

The following are quick and simple methods of obtaining azimuth.

Rough Methods of Determining Azimuth.

(1) By 'Polaris,' or circum-polar star at culmination. Compute watch time of transit, and bisect with centre wire of theodolite, at that instant, reading also bearing of R.O. on hor. arc.

(2) By a circum-polar star at 'elongation.' Compute watch time of 'elongation' and after observing the star, and R.O., add or subtract the star's polar distance from the angle observed from R.O. to star, when a fair approximation of the azimuth of the R.O. will be obtained.

The following stars are suitable for the above observations.

In the Northern hemisphere α and δ Ursæ Minoris.

In the Southern hemisphere β Hydri and β Chamæleontis.

Lastly, when ζ Ursæ Majoris is vertically *above* or *below* the pole star, the latter is at its '*lower*' or '*upper*' culmination respectively, and this can be determined with a plumb-line. This star is the second from the end of the tail of the Great Bear (the Plough). *Vide* p. 125, fig. 54.

Determination of Longitude.

Problem to be Solved.

For determination of longitude (or difference in local time at a place, from that at Greenwich), all that is necessary is to find some means of comparing the local times at the two places at the same instant.

The following are the most usual means adopted to make this comparison.

(1) Transportation of chronometers.

(2) Signal, or electric telegraph.

(3) Simultaneously noting phenomena in the heavens mutually visible, such as the eclipses, occultation, or transits of one of Jupiter's satellites.

(4) Eclipse of the sun or moon.

(5) Moon culminating stars.

(6) Lunar distances.

The calculations involved in most of the above are too lengthy to treat of here, but the following general description of each method may be of use.

(1) The chronometers (a number are generally used) having been very carefully rated, and compared with the local time at the reference meridian station (such as Greenwich), are carefully transported to the other station, and again compared with the local time at that meridian, when the mean difference gives the 'difference in longitude.' 50 chronometers were used in the years 1849–55 in determining the difference in longitude of Cambridge in the United States, from that of Liverpool, the *mean* obtained by the voyages in each direction, only differing by 0·15s. of time.

(2) This is the most accurate method, when available, the comparisons of time being readily and conveniently made by signals transmitted by the operators. The mean of a large number of comparisons is adopted.

(3) There are three different methods here possible :—

(*a*) Eclipse of satellite by the planet.
(*b*) Occultation of satellite by the planet.
(*c*) Transit of satellite, or the shadow of the satellite, over the planet.

As the N.A. only gives *approximate* times for the phenomena in (*b*) and (*c*) these latter are of little practical use, but under favourable conditions and with a good telescope the observations of an *eclipse* (*a*), may give valuable results.

This observation (*a*) has the great advantage of requiring no computations. Its practical difficulty, however, is in determining the *exact* moment of the disappearance (D), or re-appearance (R) as the satellite disappears from, and re-appears into view only *gradually*, and not instantaneously.

(*a*) *Eclipse of Satellite.*—The Greenwich time of the eclipse (D) of each of its satellites by the shadow of the planet Jupiter, and of their re-appearances (R), are recorded in the N.A. These phenomena being independent of the earth are *simultaneous* for all observers, in whatever longitude they may be. It requires a powerful telescope to ascertain the exact moment of the eclipse, but if the disappearance and re-appearance are both observed with the same telescope, the mean

of the two times will give a fair result. Observers who wish to derive their difference in longitude by these eclipses should use telescopes of the same power, and observe under the same atmospheric conditions, as far as possible.

The difference between the *correct local time* of the eclipse, and the time given in the N.A., gives the longitude of the place (from Greenwich) without any further computation.

(*b*) *Occultation of Satellite.*—The *approximate* Greenwich times of the disappearance behind the disc, or *immersion* (Oc. D.) and the re-appearance, or *emersion* (Oc. R.) of each satellite, are given in the N.A. The predicted times serve only to enable observers to direct their attention to the phenomenon at the proper moment.

(*c*) *Transit of Satellite over Jupiter's disc.*—The *approximate* Greenwich time of 'ingress' (Tr. I.) and 'egress' (Tr. E.) or the first and last instants when the satellite appears projected on the planet's disc, are given in the N.A.

The Greenwich time of the *transit of the shadows* of the satellites are also given, and can be also used in a similar manner.

(4) The necessary computations are very laborious when calculating longitude by the method of eclipses of the sun or moon.

(5) This method may be described as follows.

The meridian transits of the moon, as well as of certain selected stars (transiting about the same time) are observed, from which the local time of transit of the moon can be accurately determined. From the N.A. the R.A. of the moon at Greenwich, at the same instant as the local observation, can be ascertained. The difference in the results obtained gives the change of the moon's R.A. due to the 'difference in longitude.'

A change of two minutes in the R.A. of the moon is produced by about 15° of longitude.

(6) The calculations for longitude by sextant measurements of the angular distance between one of a *selected* set of stars, and the illuminated disc of the moon, a method termed 'lunar distances' are very complicated, owing to the proximity of the moon to the earth, and for the study of this method the student is referred to Chauvenet's or Loomis's works on Astronomy.

After the most laborious calculations, the results are so untrustworthy that even navigators (who formerly placed their sole trust in this method) very seldom use it nowadays, preferring to trust to their chronometers to find the difference between local and Greenwich time at the same instant.

Graphical Methods of finding Time and Azimuth.

Time and Azimuth found Graphically.

Time and azimuth can be found very simply, and with a fair degree of accuracy, by spherical geometry, as follows.

Describe a circle to any given radius (the larger the better for the sake of accuracy), and let it first represent the plane of the celestial meridian at any given place. Draw lines Hs HN and Z N, fig. 60, perpendicular to one another, and representing the rational horizon and the plumb-

line including the zenith and nadir. With a good circular protractor, lay off the angle E O H, equal to the co-latitude of the place, and produce it to Q, so that E Q represents the plane of the equator, and draw the polar axis P_N P_S through the centre O, and perpendicular to E Q.

Let the stars observed, *rising* in the east and *setting* in the west, be represented by S_E and S_W respectively, having declinations δ_E and δ_W at the time of observation.

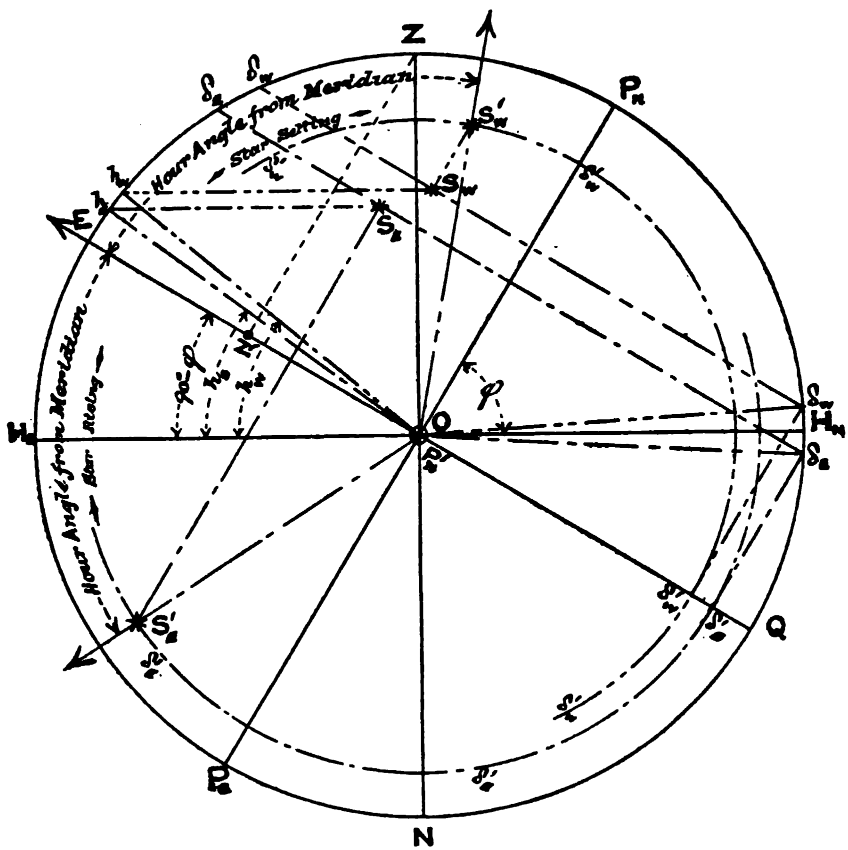


FIG. 60.

Lay off the angles Q O δ_E and Q O δ_W equal to the declinations, and through the points δ_E and δ_W, draw lines δ_E δ_E and δ_W δ_W parallel to the equator E Q.

These lines will represent the 'traces' of the daily paths of the stars S_E and S_W. Protract the reduced altitudes h_E O H and h_W O H, and draw the lines through h_E and h_W parallel to H_S H_N representing the traces of circles of equal altitudes. It is now obvious, that the stars at the time of observation were at the points S_E and S_W, being the intersections of each pair of traces of 'altitudes and declinations.'

So far the procedure is the same, whether the stars (including the sun) be observed for time or azimuth.

Construction for Time.

If for time, the hour-angle is found as follows. Imagine the semicircle E P_N Q, fig. 60, to be turned up till P_N takes the place of O so that we are, as it were, looking down on a hemisphere having the pole in the centre and the surface of the paper representing the equatorial plane. By drawing perpendiculars through δ_E and δ'_W to δ_E and δ'_W on the line E Q, and describing circles with O (now P'_N) as a centre, and the intersections of these perpendiculars with E Q as radii, we represent the paths of the stars S_E and S_W as seen in plan. Through S_E and S_W, draw lines parallel to P_N O P_S

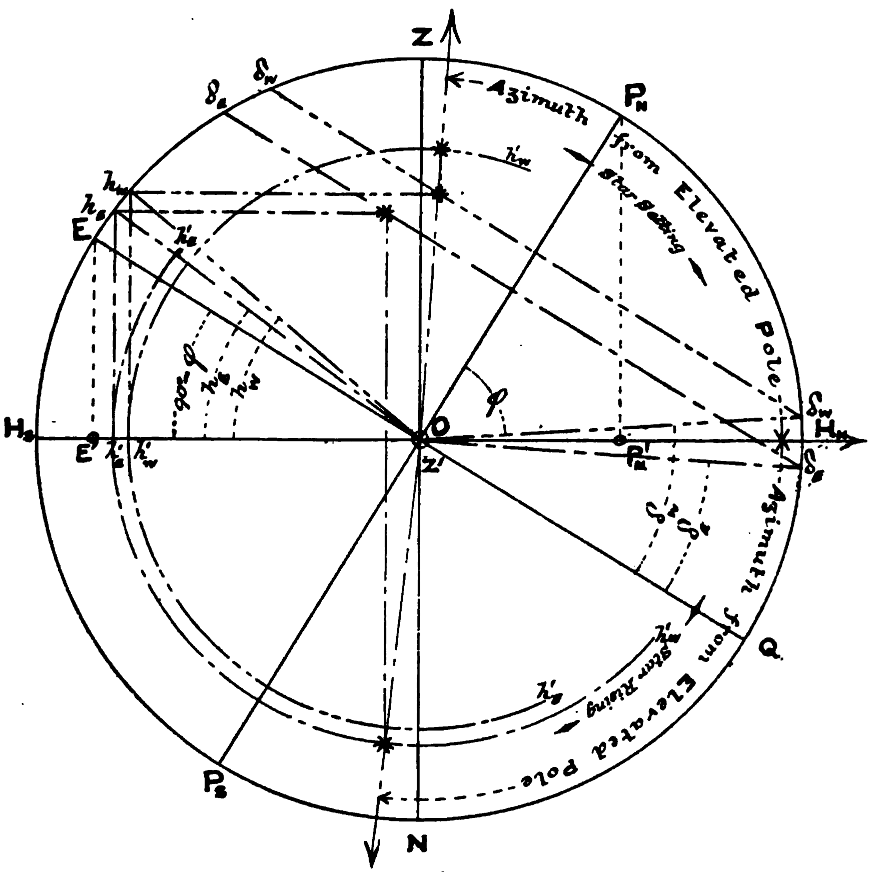


FIG. 61.

meeting these paths of the stars or declination circles in S'_E and S'_W respectively. It is clear that S'_E and S'_W now represent the positions of the stars on the hemisphere, in plan, and that if lines be drawn through them and the centre O, that the angles S'_E O E and S'_W O E so defined are the hour-angles measured in arc from E O the trace of the meridian.

Construction for Azimuth.

If for azimuth, the azimuth angle from the elevated pole is found as follows. Imagine the semicircle H_S Z H_N, fig. 61, to be turned up till Z takes the place of O, so that we are, as it were, looking down on a hemisphere having the zenith in the centre, and the surface of

the paper representing the plane of the rational horizon. By drawing perpendiculars through h_E and h_W to h'_E and h'_W on the line Hs O and circles with O (now Z′) as a centre, and the intersections of these perpendiculars with Hs O as radii, we represent the circles of equal altitudes of h_E and h_W respectively as seen in plan. Through S_E and S_W, draw lines parallel to Z O N meeting these circles in S'_E and S'_W. It is clear that S'_E and S'_W now represent the positions of the stars on the hemisphere, in plan, and therefore the inclination of lines drawn through them and the centre O, to the plane of the meridian, viz. S'_E O H_N and S'_W O H_N measured in the plane of the horizon, will give the azimuth of the stars, respectively, from the elevated pole at the time of observation.

The above geometrical solutions are very useful as a check on clerical errors, which may creep into analytical solutions.

It will be understood, that with a circle of 10 to 12 inches in diameter, a metal protractor reading to minutes of arc, and a fine-pointed hard pencil, considerable accuracy is attainable by the above method.

A General Graphical Method.

In one of the writers' opinion the Astronomical Triangle can be solved even more readily by the "unwrapping" conception used in Chapter VI., when discussing Spherical Trigonometry. For example, given zenith-distance, polar-distance and co-latitude, find hour-angle and azimuth. In the figure (page 76), lay off

$$\angle C_1 O B = \text{zenith-distance or co-altitude.}$$
$$\angle B A O = \text{co-latitude.}$$
$$\angle C O G_2 = \text{polar distance or co-declination.}$$

Complete the construction, as described.

Then

$$\angle C_2 M C_4 \text{ or } C M C_4 = \text{azimuth.}$$
$$\angle C_1 N C_3 \text{ or } C N C_3 = \text{hour-angle.}$$

$$\left.\begin{array}{lll} \text{In short, for C} & \text{write} & \text{S} \\ \text{,,} \quad \text{B} & \text{,,} & \text{Z} \\ \text{,,} \quad \text{A} & \text{,,} & \text{P} \end{array}\right\} \text{throughout with subscript numbers.}$$

It will be seen that every case of spherical trigonometry can be solved graphically, by this one principle of construction. It has moreover the great merit that one can see at a glance whether the spherical triangle is "well-conditioned" or otherwise. If the construction lines meet in very obtuse or very acute angles, so that the point of intersection is vague and badly determined, then the spherical triangle is ill-conditioned, and high accuracy cannot be expected even by logarithmic computation. More accuracy is attainable by this graphic method than might at first sight be supposed. The writer remembers having obtained the longitude of a ship, within 4 miles of that obtained by the captain, by computation, using the captain's corrected observations, and he may add his drawing-instrument, by no means instruments of precision. The writer believes that this method of construction will be of use to the learner, as it will enable him to see exactly what he is doing by computation. It will also be of use in solving subsidiary problems, such as the determination of a star's hour-angle when on the Prime Vertical.

CHAPTER X.

ROUTE SURVEYS AND RECONNAISSANCE.

General Remarks.

ROUTE surveys may be made, and reconnaissances conducted, in many ways, varying in degree of accuracy and completeness from the roughest sketch, under unfavourable circumstances, to a near approach to the finished work of a regular survey, where the most careful methods can be employed. The objects to be attained by the survey, the nature of the country, the strength of the party, and the time allowed, all combine to determine the methods to be adopted, and the amount of accuracy, detail, and finish possible. It may be proposed to construct a road or a railway, through a more or less unknown and unexplored tract, perhaps uninhabited and jungle-covered, where the nature and extent of the country to be examined would preclude rigorous operations, till the probable direction the line would take be ascertained. To avoid unnecessary labour and expense, a preliminary reconnaissance would have to be made, to determine the most favourable run of the proposed work, and to decide as to the portions which might with advantage be more accurately surveyed. In open, favourable country, this preliminary reconnaissance may be made with the plane-table only, and yet considerable accuracy obtained. In low forest-clad country, route surveys must be carried through various portions following the existing tracks, the courses of streams, &c. and with care, thought, and observation, on the part of the surveyor, he should be able to give a very fair idea of the country passed through. The story is well known of the great painter who, on being asked what he mixed his colours with to obtain certain effects, replied, 'with brains,' and if a surveyor supplement his equipment with brains, he will find that excellent results can be obtained, by whatever method and whatever instruments he employs. Few investments give such a large return, as the outlay of painstaking labour and thoughtful care by the surveyor, at the commencement of his work, and much trouble, worry, and confusion, will thus be saved to himself, and to everyone employed. The writer would here allude to the necessity for a surveyor (who is to carry out a reconnaissance, or an exploration) to make himself thoroughly familiar, before starting, with the instruments to be employed, and with the various astronomical observations and computations which will be found necessary.*

Rapid Triangulation.

The safest basis for all survey work is, of course, triangulation, and rapid triangulation, though not so accurate as regular operations, can (more often than is generally supposed) be

* This section was written by the late General Woodthorpe, R.E., C.B., some time Surveyor-General of India, and embodies his ripe experience in this class of work.

carried out with satisfactory results—as has been done, for example, in the boundary operations in Afghanistan, and on the borders of China and Siam.

Such triangulation always proves most useful. It enables the topography to be executed with accuracy with the plane-table, by interpolation from the triangulated points, and the positions of these points being obtained by computing their latitudes and longitudes, they can be plotted easily and accurately to any scale, on any sheet. Error is *greatly* eliminated, and the sketches made by different men, even though they do not overlap, or adjoin each other, can be placed in absolutely correct relative position in the general map, the compilation of which is thus greatly facilitated. Distant points can moreover be fixed with sufficient accuracy, and heights correctly determined, thus affording throughout the work, valuable checks and standards of comparison for the less elaborate methods, and rougher instruments.

In triangulating for frontier surveys, reconnaissances, &c., smaller instruments are used than on regular work, the bases cannot be measured with the same refinement, there is less regard for symmetry in laying out the triangles, and the three angles of each triangle are not always observed. It will often be found impossible to visit hills beforehand, to clear them and erect signals, so that natural marks such as rocks, trees, etc. must be observed to.

Theodolites used in India for Frontier Work.

In India, the 6″ transit theodolite is found to be the most useful for frontier work. It is fitted with a 'micrometer eyepiece' for subtense work, and with the level fixed to the vertical vernier bracket and *not* to the telescope. This theodolite is packed in two boxes, for convenience of carriage in hilly countries, the horizontal circle in one box, and the vertical circle in the other. Occasionally, in climbing very steep and difficult peaks it is necessary to dispense with boxes, and to carry the various parts of the instruments carefully wrapped in thick cloths and tied tightly on to men's backs. Where it is difficult to carry a 6″ theodolite, a 3″ has been found useful, and when carefully manipulated gives good results for triangulation and astronomical observations. It has been found in practice that the tangent and clamping screw heads, as usually supplied, are too small for night work, being difficult to find readily, and it is recommended that this instrument be fitted with larger ones. The stand of the theodolite is a folding tripod in which the heads of the legs are capable of being screwed so tightly that the whole stand becomes absolutely rigid, and the feet should be so shod that the shoes cannot work loose and cause unsteadiness. In unstable ground, steadiness may be secured by driving in three wooden tent-pegs nearly flush with the ground and resting the stand on them.

Base Measurements.

The great trigonometrical and topographical branches of the survey of India have now so far extended their operations that frontier work may generally be started from some well fixed points affording accurate bases. This is not, however, the case everywhere, and in many places bases must be measured, from which to commence triangulating. The base will generally lie in a valley or on low ground, and may be measured by a steel band or a well tested chain, or by some subtense method, should the ground be too broken or rough for chaining. A base was required during the

Afghan campaign in the Kuram valley, which is a large, open, and treeless plain. It had to be measured over very stony ground, much cut up by deep and wide ravines. A 6" theodolite and a 10-foot bar were employed. Each reading was taken several times very carefully. The length of the base was 4 miles, one end being on the towers of the fort, which raised it 30 feet, and when it was afterwards connected with the triangulation brought up from India it was found to differ from the correct value by 3 feet only. A small base over uneven ground has been measured with a 10-foot bar, or bamboo, supporting the ends on plane-table stands carefully levelled and centred. This, though giving fair results, is a tedious and lengthy business. While the ground for the base is being prepared, if circumstances permit, a short traverse might be run with a subtense instrument, a perambulator or even by pacing, from which the positions of the surrounding hills could be cut in on the plane-table. The surveyor will then be able to lay out his triangulation with greater regard to symmetry than if this precaution were neglected. When possible the position of the base should be so selected as to afford the greatest facilities for extending the triangulation symmetrically, or an error in the base might throw the work out very considerably.

It is seldom that ground can be found, on which to measure a base more than 1 or 2 miles in length, but for any extensive survey it should not be less than ½ mile, then if this measured base be too small, or not conveniently situated with reference to the neighbouring peaks chosen for extension, a larger base may be obtained from the measured base by triangulation. (*Vide* p. 183, Part I.)

Station Marks and Signals.

When the hills for forward stations have been decided on, and while the base measurements are being executed, men should, if possible, be sent ahead to clear the summits, when forest-clad, and erect signals on the highest points. In Afghanistan this was seldom necessary, for besides the hills being as a rule bare and rocky, their summits were generally marked by a cairn of loose stones, erected in pious memory of some saint. Occasionally there were several such cairns on a peak, and then they were all observed and the most suitable one chosen when the hill was visited. The cairns were easily removed to make way for the theodolite, and were carefully rebuilt when the observations were concluded. Before removing the cairn its centre should be noted. This may be done by laying a couple of lines of stones whose prolongations intersect in the centre of the cairn.

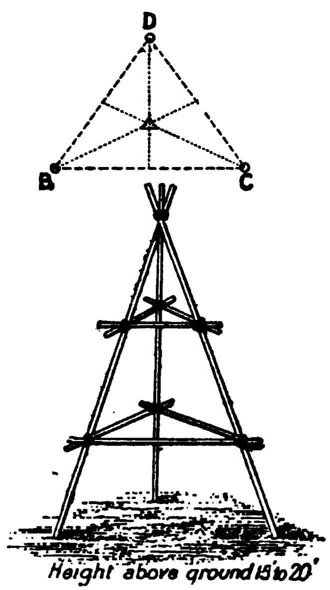


FIG. 62.

On the North-east Frontier of India and in Burma, where bamboos are obtainable everywhere, the following signal is invariably used.

Let A fig. 62 be the station, and the centre of an equilateral triangle B C D

described around it. At B, C and D, holes are dug 3 or 4 feet deep sloping upwards towards A. These holes receive the butt ends of long poles which are tied together at the top and braced at the sides. While this is being done, men are busy preparing mats of bamboo, with which the sides of the tripod are to be covered to within 6 feet of the ground. A basket in the shape of a double cone connected at the bases is also formed by splitting a bamboo up for about 3 feet of its length at the smaller end and working the latter into the required shape by means of strips of cane or bamboo. A long narrow flag may be advantageously used in place of the basket. This basket or flag is firmly secured to the tripod and centred over the station mark, which may be a circle inscribed on a rock *in situ* or on a stone sunk in the ground. The inside of a bamboo is white, and retains its brightness for some time when split. This side is placed outwards in the mats and basket, and the mark shines brilliantly in sunlight like silver. It is visible for very long distances, and sometimes when the hot weather has set in, such signals can be made out several miles away, glistening in the sunshine when the haze has blotted out the hills on which they stand. A single tree should always be left standing near the highest point when a hill is cleared, and its position with reference to the station carefully ascertained and noted in the 'angle book.' This assists the observer in recognising his station, and should the signal be blown down, or destroyed by wild animals or otherwise, the tree can be observed to. In very exposed situations, a couple of trees or even a small clump may be left. Sometimes, when a hill is very narrow and rocky, it may not be possible to erect a signal, and in such cases two trees should be left standing and both observed to. When a hill has not been cleared, or marked in any way, it is best to observe to the highest point or to any conspicuous objects, bushes, rocks, etc., but these may not be recognisable from other hills, or even when visiting the hills on which they stand. If several points in the same hill are of the same height, or seem equally suitable for stations, they should all be observed. If a surveyor cannot communicate with others working near him, and has reason to suppose that certain hills have been utilised by them as stations, though no signals may be visible, every point on each hill, on which the theodolite could have been set up should be carefully observed. By following this practice during the Afghan campaign, when intercommunication between the Khyber and the Kuram columns was at times impossible, the triangulation carried on by the surveyors with each column was connected, and eventually computed satisfactorily, the correct angles being found during the computations by elimination, or by consultation among the observers, when they were able to compare notes.

Lengths of Sides.

The lengths of the sides of the triangles vary according to circumstances, such as the height of the hills, their configuration, the clearness of the atmosphere, the nature of the signals or objects observed, and the maintenance of symmetry, as far as possible. As a rule, the sides are from 10 to 20 miles in length. In Afghanistan it was possible to have triangles with 30 to 40 miles sides, and peaks were intersected at a distance of 100 miles. This was also possible during clear weather in the Shan States on the borders of Burma and China.

Observations.

In observing from a station, if time is limited, the surveyor must use his judgment as to what to omit, remembering always

that the first object of the triangulation is to assist the topography. A good rule is to observe everything that is observable. It is impossible to say what will not be useful, and even a single ray to the most distant peak, village, or tree may prove very valuable as a check on topography to keep it in its place, and if possible all these rays should be laid down on the plane-table. It is sometimes the practice to set up the theodolite by the compass, so that the telescope points to the magnetic north when the horizontal limb reads 360°. By this means all readings are also magnetic bearings, and it frequently assists the finding of a point which has been observed, to know its magnetic bearing from a previous station. Moreover, unless the bearing of even so plain an object as a heliograph, is known, it is possible in hazy weather, or when the carelessness of the heliographer prevents a full light from reaching the observer, to pass it unnoticed in moving round the telescope, however slowly.

Another advantage claimed for making the telescope point to the magnetic north when the theodolite reads zero, is that the variation of the compass at any station may be found by comparing the computed azimuth of any ray with its magnetic reading.

Vertical Angles.

For vertical angles both verniers should be read, with at least one change of face *in all cases*. Horizontal and vertical angles to stations should *never* be observed simultaneously. Vertical angles to all stations more than four or five miles distant should be observed at about the time of minimum refraction, say between 2.30 and 3.45 P.M. For intersected points the time is of less consequence, and so long as the observations are not made in the early morning or late evening hours, the observer may use his own discretion about adhering to the rule. If vertical angles be observed at an hour far from that of minimum refraction, observation should be made to some known station at the same time, for the height of this station being known, the refraction, at the time of observation, can be computed by means of the vertical angle then taken to it. When possible, observations should be reciprocal, i.e., if A has been observed from B, B should be observed from A, as the effects of refraction are thereby eliminated, and a co-efficient of refraction obtained for reducing single vertical observations. The height of the instrument and signal above the station platform, must always be recorded in the 'angle book' and in observing vertical angles to intersected points the particular part of the object intersected should be noted, as well as its height above the ground level.

In observing to the tops of trees, or to signals in trees, the level of the ground is the height required, and if it can be seen it should be observed also in the case of intersected points. It will be impossible otherwise to measure the height of the tree, but from an observation this may be computed and the correction applied to all other vertical observations, from stations where the base of the tree is not visible. All circumstances likely to affect observations should be noted against each angle in the 'angle book,' also where any angles are exceptionally good or doubtful. This will assist the computer in apportioning the triangulation errors.

Recognition of Stations, Peaks, etc.

It is a good plan, in an unknown country especially, to make a panoramic sketch of the intersected peaks as they appear in the telescope, marking carefully their outline and any peculiarity for identification. This sketch may be made in the 'angle book' or on the plane-table. The peaks are numbered in the order in which they are observed from the first station, and the numbers adhered to throughout. Names are not always obtainable, but when they are, they can be entered against the numbers for use in the final map. A triangulation chart should always be kept up, either stretched on a plane-table or capable of being pinned to one. This chart will greatly facilitate the work, and enable the surveyor to identify peaks which he would otherwise find it difficult to recognise. Mountains assume such wonderfully varying outlines from different points of view, and a broad square hill seen from the front will often become a sharp cone in profile, e.g. the rock of Gibraltar. Again, a hill presents quite a different appearance when viewed from below to what it did when seen from neighbouring hills, but when fixed on the plane-table there is no doubt about a peak. When, therefore, points have been observed to with the theodolite, the earliest opportunity should be taken of fixing them on the plane-table. If one or more rays have been taken to a hill on which a surveyor finds himself, but which he is not quite able to identify, he can ascertain his position by re-section, and if it falls on a previous intersection, or even on a single ray, it is probably correct. All other points around can then be identified. Again, suppose A and B (fig. 63) are two fixed stations, and from A two hills, C and D, have been observed, but it seems doubtful if they can be readily recognised from B. It would probably be easy to find a plane-table station near A, say E, from which to fix C and D with sufficient accuracy to ensure recognition from B.

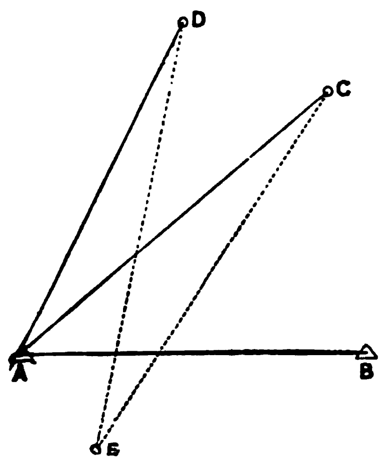


FIG. 63.

Description of Stations in Angle Book.

All stations of observation must be very carefully described in the 'angle book' at the end of the observations made at the stations. This description of each station includes, the situation of the hill, the exact position of the station mark with reference to the single tree left standing, or some other well defined natural object, the directions and distances of neighbouring villages, the local name of the spot (if any exist), the village in whose lands it is situated, the best way of reaching it, and any other information likely to assist any one subsequently wishing to make use of the station. All this is very necessary, especially in forest-clad districts, where the villages are constantly shifting, and where the jungle growth speedily removes all trace of village sites, fields and paths, even as though such had never existed.

Stations in forest-clad country.

Forest-clad country such as is found on the N.E. frontier of India and Burma, the forests of the Gold Coast etc., presents the greatest difficulties to the surveyor, and considerable expenditure of time and money and much labour are required to clear the hills for stations. In such cases the stations selected should, of course, be as few in number as possible. Sometimes, as in parts of Assam, the low undulating country is of such a general dead level, that little is gained by clearing, or the hills are so broad and flat that complete clearing is out of the question. It then becomes necessary to raise the theodolite above the level of the surrounding country, sufficiently to obtain a clear view with a small amount of lopping of the branches of the neighbouring trees, and perhaps a narrow ray cut here and there to stations. The following method was employed by a survey party working in Assam. A suitable tree was selected, its trunk cut off at a convenient height 30 to 45 feet above the ground, and a scaffold erected around it. On this, and at about $4\frac{1}{2}$ feet below the top of the trunk, a platform of bamboo was made for the observer to walk on, the platform and scaffold being quite isolated from the tree-trunk. The theodolite, a 6-inch, was set up on the tree-trunk, dispensing with the stand. The highest point at which the theodolite has been used in this way, with perfectly isolated platform, was 45 feet. Some cotton trees* with bare straight trunks rise to 80 feet, and platforms have been constructed at that height in these and other trees for plane-tabling, but they are not steady enough for theodolite work.

Action, should triangulation break down.

When triangulation has been carried some distance, there is a chance of its breaking down, owing either to the configuration of the country, the non-identification of previously observed points, or circumstances which prevent flanking stations from being visited. When this is the case, if a station is visited before 9.30 a.m. or after 2.30 p.m., an azimuth of the sun, if visible, may be observed and referred to one of the back stations. This azimuth, with the observed angle between any two well fixed points, not too much in line, will give a very fair fixing of the position of the visited station, from which other points may then be observed. Interpolations play a more important part in rapid work than in regular survey, and stations may be made in the middle of a valley where there is no conspicuous object to which it is possible to observe.

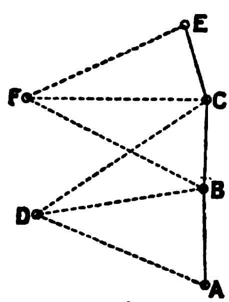


FIG. 64.

Suppose A and B (fig. 64) are two fixed points on a range parallel to the route being followed, D being a camp on that route, and that a peak C on the range A B has been observed from B. The distance A B and azimuth are known. The surveyor observes the azimuth of A or B from D, and the angle A D B. With these data, the other angles of the triangle A D B may be found, and the position of D computed. From D, C may be observed and fixed. C must then be visited and another forward hill E observed.

* Probably the Silk-cotton Tree, common in the West Indies, for the ordinary cotton plant is a mere shrub.

From C and a further point F on the route, the hill E is similarly fixed; and so on, thus triangulating along a single ridge, but without observing two points on the flanks. With care, this method is susceptible of considerable accuracy.

Again, let B, C, D, E, (fig. 65) be points along a continuous ridge, and *a*, *b*, *c*, *d*, peaks on the flanks which cannot be visited, A B the last base fixed by triangulation, from which points *a*, C, *b*, have been fixed. Proceeding to C, angles are observed to all the points. Then in the triangles C B *a*, C B *b*, we have the base C B and the two angles at the base in each triangle.

We can then find the third angle, and determine the bases C *a*, C *b*. Visiting D we observe to C, *a*, *d*, E, *c*, and *b*. In the triangles C D *a*, C D *b*, we have the angles at C and D, and can therefore deduce those at *a* and *b*, and with bases C *a*, C *b*, we get values of C D, from which base, *c* and *d* are fixed, and so on.

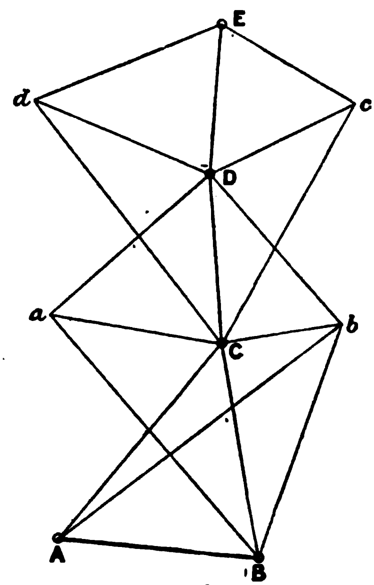


FIG. 65.

Action, when Triangulation is interrupted.

If the triangulation be interrupted, a careful traverse checked by astronomical observations should be carried on till the triangulation can be started again, when, from a fresh base, it may be possible to connect the new work with the old. If, for instance, a pass has been crossed, and the surveyor finds himself unable to see more than one fixed peak (X) and cannot ascend any hills in the neighbourhood, he could measure a base and observe to X from this base, and also the azimuth of X from either end. The values for the base can then be computed, and fresh forward triangulation started.

When the route approaches north and south, latitudes should be observed at all camps, and azimuths observed to all peaks in the line of march, or nearly so. When camping abreast of any of these peaks, their position must be fixed with reference to the camp, by means of a short base with the usual latitude and azimuth. The difference of longitude can then be easily computed.

A Series of Azimuths Necessary in Rapid Triangulation.

Of course, in all rapid triangulation, as will have been gathered, a continuous series of azimuths is essential, and this involves the computation of the latitudes and longitudes of all stations and intersected points. Computations should always be kept up to date if possible. If the latitudes and longitudes of points required for use are calculated, these points can be more accurately plotted by their co-ordinates than by distance and bearing. The distance between points not directly connected can be readily computed, and this is particularly useful for interpolations. Again, latitudes and longitudes are easily communicated to other surveyors by telegraph, heliograph, or letter.

Triangulation and Topography by One Surveyor.

If triangulation and topography have to be performed by one person, he should consider carefully how much attention to bestow on each. The objects of the survey will assist him in coming to a conclusion. Triangulation by itself is useless, and unless it is necessary to know the positions of certain points with accuracy, or unless the triangulation is intended to assist other topographers, it should be subordinated to the topography, and carried out only so far as it assists the execution of the work actually in hand.

Heights in Plains and Valleys.

Some surveyors neglect to fix any heights in the plains or valleys, only observing to hill-tops. Of course, the heights of as many hills as possible should be fixed, not necessarily to be entered in the final map, but to assist the topographer to interpolate heights in valleys and open ground. But it is also important to observe vertical angles to well-marked points of features in low ground, such as conspicuous bushes or rocks on the banks of a river, the junction of two streams, pagodas, temples, the most prominent house in a village, etc. The distances for computation may be taken from the plane-table. It is impossible in maps on a small scale, to give, by any system of hill shading, an accurate idea of the relative heights of neighbouring valleys with intervening ranges, but in consulting maps, whether for movement of troops or engineering operations, one is materially assisted in coming to a right judgment, by a few figured heights in the low ground, as well as on the range between. Again, passes should always be observed to. It is often more important (though this is not so often realised), to know the height of a pass or the saddle between two peaks, than the heights of the peaks themselves.

Height of Base Measurement for Reduction.

In measuring a base, if there is no reliable height for it, an approximate height must be obtained from barometric observations and used throughout the triangulation. When eventually a trustworthy value for the datum is arrived at by connection with a trigonometrical series, or to the sea, etc., a constant correction can be applied to all the heights. In Afghanistan, barometric heights were found to be very unreliable, even those obtained from a George's mercurial barometer being unsatisfactory. It was found that the temperature correction as given in text-books is excessive, and it is suggested, that this is because in that country the temperature observed at the ground level, is due to a highly heated stratum of air, giving no clue to the temperature of the column of air, immediately above the place of observation.*

* To obtain a difference of altitude by the barometer, the density of the air-stratum between the lower and upper stations must be known. As the density varies, inversely as the temperature, the mean temperature of the intervening air-stratum must be known. In the formulæ ordinarily given in text-books, the mean temperature of the intervening air-stratum is assumed to be the arithmetical mean of the air-temperature at the upper and lower stations respectively (mean of detached thermometer-readings). In most cases there is no other course open but to make this assumption, but it may lead to erroneous results. Possibly the anomalies observed in the case of Afghanistan, may have been due to this as well as to the cause cited in the text. High accuracy cannot be attained by means of the barometer, however accurate the instrument, except under favourable conditions, such as the determination of the height of a

Among aneroids, the smaller patterns will be found to be quite as accurate, and much more convenient to carry about, than the larger. Very satisfactory results have been obtained from boiling-point thermometers, if three at least are used, and the mean taken. Aneroids should always be read immediately on arrival in camp and again next morning before starting, the changes due to climatic variations of pressure being thereby eliminated to some extent. During halts a series of careful hourly readings should be made to get data for correcting observations for the diurnal wave of pressure. Whatever instruments are used, no opportunity should be lost of comparing the results with the heights obtained by triangulation. Such valuable opportunities occur while waiting on a hill for it to be cleared, or while observing, and barometers may be hung up in a convenient spot and read at intervals, while observations are also made with the hypsometer.

Watches. Watches form an important item in an explorer's outfit. Chronometers are unsuitable for work on land, as they do not stand even the most careful method of carriage. Keyless half-chronometers have been found to give exceedingly good results. An explorer should have at least three of these with him for comparison, and care should be taken to keep them always under the same conditions of position, temperature, etc. Many watches vary greatly in rate when changed from a horizontal to a vertical position, when lying on their face or the reverse, or if the ring is to the right or left of the watch. Jolts should be carefully guarded against, and a vertical position may be secured by having special pockets in a coat and keeping the watch hung up in the coat at night. Great changes of temperature should also be avoided by wrapping up the watches in cloths. It is well to carefully pack your watches yourself, and the writer has found a despatch-box very suitable as it could be fitted into a light bamboo sort of chair, and carried on a man's back. The box was simply lifted from the table into its chair, and back from the chair to the table, and the watches always remained in the same position either on the march, or when at rest.

Before starting, the rates of the watches should be ascertained as nearly under the conditions of travel as possible. During halts, the rates should be checked by observations for time, and the errors determined whenever the longitude of a place is known.

Precautions should be taken against damp, getting wet in crossing streams, or accidents, by adapting waterproof covers, or by waterproofing the case as described in "Hints to Travellers."

tower, by simultaneous observations at top and bottom, owing to the impossibilities of ascertaining the mean temperature of the intervening air-stratum. One of the writers believes that had General Woodthorpe been acquainted with the Wagner level, described in Part I. page 85, and realised the rapidity and accuracy attainable by its use, he would have adopted it for ascertaining the levels of camps and stations along the line of route in lieu of the barometer. The writer has levelled with this instrument a circuit of 3½ miles over rugged ground in 4 hours, with a closing error of 0·30 foot. He could guarantee that an observer and two staff-holders could easily keep up with the march of an exploring party and ascertain the levels of stations along the route, with far greater accuracy than by means of the barometer.

CHAPTER XI.

PLANE-TABLE TOPOGRAPHY.

The Plane-Table.

THE plane-table is without doubt the most accurate topographical instrument we have, and one of the most useful to explorers. It enables the triangulator to recognise his points, and when necessary, graphic triangulation can be carried on successfully side by side with the topography. For instance, two officers of the Indian Survey Department working in 1872–3 on the North-East Frontier of India, were able to carry a series of triangles with their plane-table through the Naga Hills to Manipur, erecting the usual marks (but without astronomical checks) during a rapid march of 120 miles, and when this work was afterwards incorporated in the regular triangulation, very little correction had to be applied anywhere, the last station in Manipur being only out of position by 2300 yards in longitude, and less in latitude. It was done by each taking a side of the series daily, and besides plane-tabling, these officers managed to observe a number of angles both of the triangles and intersected points, the computations being worked out after they reached Manipur.

The plane-tables in use in the survey of India are similar to those described in Chapter VI., Part I., being of a simple and strong construction. The general size is 24 by 30 inches, although on some of the Frontier expeditions, a smaller and lighter one, 20 by 24 inches, was used with advantage. The writer has seldom used the latter, as he has found it possible, in most cases, to carry the larger size—the advantages of which are obvious, especially if the operations are likely to be extended over an area greater than the field of the small board. Sometimes, also, it is not possible to foresee exactly in which direction the work may eventually lie, and a large board enables the surveyor to face a change of front with equanimity. Again, on a large sheet, very much may be done in the way of tentative geography, "guesses at truth." The writer has always sketched in lightly, in pencil, what seemed to be the probable geography of all distant country, embraced by his board, and beyond the limits of ground which could be accurately delineated. He has found it exceedingly interesting, as the work progressed, to watch these guesses turn into facts, or, if wrong, to see where, how, and why he was misled. It is an excellent education in reading country, and even a "guess at truth" is always better than an absolute blank on the board.

Suitable scales.

The scale to be adopted is decided by various considerations, such as the objects of the survey, the amount of detail required, the area embraced, the time allowed, etc. For trans-frontier surveys and explora-

tions, the scale adopted by the survey of India is 1 inch = 4 miles for general work, special portions being surveyed on a larger scale where necessary. On the small scale maps, portions of country may be generalised which would have to be altogether left blank on a larger scale map requiring more attention to minor features, and distant triangulated points may be plotted which materially assist the plane-tabler. The engineer in charge of the operations, if he had a large party, would find it a good plan to have a small scale board embracing the whole area of operations, on which he could note the progress of the work, compiling it from the tracings sent in by his assistants, and checking it in the field from time to time.

The methods of 'setting up' the plane-table and finding the surveyor's position thereon from certain known fixed points have been fully dealt with in Chapter VI., Part I. These are the same, in principle, for all plane-table work, and only a few remarks on this head are here necessary. Before beginning a reconnaissance, a surveyor should (when triangulated points form the basis of his work), start it from the nearest station, setting up and levelling his table over the mark stone. In hilly country it is very necessary that the table should be level, for if it is not, where some points tower above the observer, while others lie far below him, the fixing will be inaccurate. The board is now oriented by placing the sight-rule on the line joining his station with the most distant visible fixed point plotted on the board, and turning the table round in azimuth till the sight-rule intersects that distant point. All the other trigonometrical points on the board can then be tested, and if any are found to be wrongly plotted, they should be carefully compared with the computations. It may sometimes happen with intersected points, that, by some accident, e.g. two hills with similar outlines having been observed as one and the same hill, the computations give apparently satisfactory results, and a point is plotted which cannot be identified. When obviously wrong it should be at once erased, but if there be any doubt, a query may be placed against it till the question is finally decided.

The method known as 'setting by the back ray,' is the best for setting up a table in true azimuth, as it is independent of abnormal variations of the compass. The other method is by interpolation, or resection (as some prefer to call it), and this may be done by compass, and the intersection of the rays from two trigonometrical points, or without the compass, from three such points ('the three-point problem' *vide* Part I.), but in neither case is there any test of accuracy. In the first, there may be some abnormal magnetic variation, and in the second if the surveyor's position falls on, or near, the circumference of a circle passing through the three points, no satisfactory fixing is possible. It is therefore better that the surveyor should, if possible, fix himself from four triangulated points at least, trusting to his compass for obtaining an approximate position only. Positions fixed by intersection where the angle is not less than 60° may be regarded as correct, but where the angle is less than 60°, as only approximate. Of course, an excellent fixing may be obtained if several rays from A, B, C, intersect in *b* at an acute angle, and another ray from D can be obtained passing through the point of intersection and nearly perpendicular to the line bisecting the angle A *b* C, at that point (*vide* fig. 65).

Each 'plane-table fixing' should be distinctly marked with a red dot for

future reference. The knowledge of the exact spot which a man has visited and fixed is invaluable afterwards in bringing various pieces of work together, and the number and situation of these fixings afford a clue to the trustworthiness and closeness of the work. Names may be written on the plane-table as they occur, or numbers may be inserted, a list of the numbers and the names to which they refer being made at the side. Numbers do not interfere with the sketching as much as names do, especially on a small scale map, but care must be taken that they are not obliterated as the work progresses.

In order to identify newly seen features of ground and other objects, it is (as mentioned in dealing with triangulation), advisable to make a second fixing fairly close to the first and so obtain approximate positions before they change their appearance beyond recognition. This is necessary on small scale surveys, where the fixings are usually at some distance apart. A surveyor while moving along must have his eyes open, and note any changes in the landscape and especially in the appearance of objects he wishes to recognise, keeping in mind the general run and character of the country. He should learn and record the native names of these objects when he can, as his guide will be able to recognise them when he may not. If triangulated points are not visible from a position which the surveyor wishes to fix, he must interpolate it from his previous fixings, and points intersected from them.

The remarks on what to observe in triangulation, apply also to plane-tabling. The surveyor should draw rays to everything likely to be useful, as it is better to draw too many than too few, being careful, however, to avoid confusion. Not only prominent peaks and other such conspicuous objects should be noticed, but also the general direction of the valleys, bends of rivers and streams, their junctions, with all minor features, such as spurs, knolls, etc., and the points where any great changes in the gradients of slopes occur, all of which when fixed, will guide him in sketching the country. Rays to the most distant points should also be drawn, assigning to each an approximate position and making against the ray a note or little sketch of the object, such as a peak, a gap in a range of hills, a pass, a village, a temple, etc. These rays may be drawn even if the object lies off the board, making a note, near the edge, of the probable distance along the ray. It is astonishing how accurately an observant and intelligent surveyor, with practice, can estimate the distances of objects from his station, and place them in correct position on his board.

Among the mountains, where the sun is high and casts but few and small shadows, it is often difficult to distinguish under features or to separate near and low ridges from higher ranges rising behind them. At dawn or sunset, however, the ranges stand out distinct from each other, and when the shadows are long, spurs and ravines reveal themselves on what, in the bright light of day, appears to be a flat and continuous mountain face. In the Far East, white mists lie like a sea of cotton-wool over all low ground, and in every valley, from early morning till 9 or 10 A.M. These soft clouds mark out each spur and ravine, materially assisting the surveyor in following up the courses of the principal valleys with those running into them, and distinguishing between the large and small affluents of the main streams. A re-entering angle in the hills, which, during the day

seems to be only a deep ravine, is now seen to be the mouth of a long valley, and what seemed to be the opening of a wide valley is discovered to be only a short ravine.

In the early spring in the Shan States and on the borders of China and Siam, thick weather obscures all distant landscapes, and a brown haze, like a curtain, shuts out everything four or five miles away, on either side of the traveller's path. Under these circumstances survey parties are limited to plane-table traverses, working at times a good deal on 'the back ray' system, intersecting points on either side of the route, and filling in as much of the topography as is visible. Native surveyors can bring in a good deal of creditable work done in this way, the distances being generally measured by pacing. Pacing may be done very accurately, and the writer has worked with some Sepoy surveyors who could (except in very difficult ground), pace to within a quarter of a mile of the truth in a march of 15 to 20 miles. Actual traversing with the plane-table is slow work and troublesome, especially where jungle or the nature of the country prevents the path from being visible for any distance at a time. In such cases, a compass traverse may be carried on, the bearings and distances being entered in a book, and at every 2 or 3 miles, where an opportunity for sketching the country occurs, the traverse is plotted on a large scale and reduced to that of the topography, and from the position thus found on the plane-table, the topography is worked in.

Plane-Tabling, and Traversing with Prism-Compass, combined.

Where two men are available, one to traverse, the other to plane-table, *more* and *better* work can be done. When travelling from Hunza through Wakham to the Dorah Pass, the writer was obliged to do long marches averaging 19 miles a day, and continuous triangulation was impossible. He had a very good Mahratta surveyor, to whom he entrusted the plane-table, while he ran the traverse with a subtense compass and 10-foot rods as subtenses.* The variation of the compass was frequently ascertained. The Wakham valley is fairly open, a river running down the middle, and high bare hills, often snow-capped, rising on either side. The country is favourable for traversing, and distances averaging over half a mile could be measured, the rate of progress being about 2 miles an hour.

Personal Description of a piece of work done by the writer.

(*The following is given as described by the writer, the late Major-General Woodthorpe, C.B.*). 'Two men, with 10-foot rods were employed. At starting one of these men (A) was left at the camp, and walking as far as I could without losing sight of him, I set up my compass and sent the second man (B) on to the farthest point B_1 of the road visible from my station. (As a rule the subtense compass is not accurate beyond a mile, and this limited the distance along each bearing in traversing.) While he was moving there I observed back to A, took the bearing and distance, noted them in a book, and signalled to him with flag or heliograph to come away. A round of bearings to all conspicuous peaks was then taken, and by this time B had arrived at B_1. He was observed to, and dismissed by signal. He then marched on to B_2, leaving a mark at B_1 for A, a small stick with a piece of red rag attached, a branch torn from a bush,

* The subtense compass referred to was, it is believed, an ordinary compass on a tripod, provided with a telescope and vertical limb. The telescope had a micrometer eye-piece.

etc. Arrived at B_1, A halted. I had reached my second station, and B was nearing B_2. The same procedure was adopted, A and B were observed, and a round of bearings taken, A advanced to B_2 and B to B_3 and so on. Before leaving camp my surveyor set up his plane-table at A, drew rays down the valley and to all the peaks, and commenced sketching. He then hurried on to overtake me, and I plotted on his board the traverse up to that point, with the intersected peaks, which fixed his position there, and he continued his sketching. Having thus obtained his position from my traverse two or three times, he himself had intersected enough points to enable him to resect his position independently of the traverse for a time, and his work thus checked the traverse, as the traverse in its turn checked his work.' Of course, when traversing in the hills with a subtense compass, it is not necessary to keep to the road, for longer distances can often be got by going up or down the slope a little way, on either side of the path. Besides, the compass and rods are thus kept clear of the line of march, the baggage animals, coolies, etc., who otherwise much impede the work.

Use of Astronomical Observations.

As soon as the astronomical observations are completed, the latitudes should always be worked out, though this need only be done with sufficient refinement for the scale of work employed, in order to fix the position of the camp, and to check the traverse and the topography. An additional advantage in computing the latitude at once, is that if there is anything wrong about it, other stars may still be observed before it is too late. Although it is hard work and cruel sometimes, as on the Pamirs, to take stars when the thermometer stands at zero, and the observer has to warm his fingers over the candle between each turn of the screw, yet results well repay the trouble. All places where astronomical observations have been taken should be distinctly marked on the plane-table and described carefully for the use of future Surveyors.

Perambulator work.

Other means besides subtense measurements of distances in traversing have been alluded to, and these require only a few words. A perambulator is useful and convenient when the country and roads are favourable, but it must be very strong and therefore heavy, and is not very portable. A cyclist will understand the principle of the perambulator and also its limitations.

Chains.

A chain, even the lightest, is also subject to certain limitations, and cannot always be used.

Canes.

Canes were used in Assam when surveying forest-clad and swampy country, and where traversing could only be carried on along the numerous small and shallow streams. The bends were too short for any subtense methods of measurement. Fine light canes, in lengths of 50 to 100 feet and about 1·5 inch in circumference are obtainable, their length is very constant, they float in water, thus giving no trouble to the chainmen, and are easily pulled taut. They also possess the advantage over chains and ropes, that they are easily drawn through dense scrub or undergrowth without being caught by thorns.

Traversing Rapid Streams.

In traversing rapid streams, it was sometimes impossible to use canes, owing to the strong flow of the water, and frequent occurrence of deep pools.

Berthon Boat and Rate of Current.

A Berthon boat or "dug-out" (a country boat hollowed out of a single tree) may be used. Measure along the bank certain distances carefully, and then note the time taken in passing them by the boat when carried down by the current only. This gives its rate, and a prismatic compass, suspended in gimbals, standing in the boat, like a ship's compass, gives the bearings, and thus with a watch it is possible to make a rapid traverse of a stream with all its windings. With a fixed point to start from, and another to close on, this is a sufficiently accurate way of filling in such detail, especially as such streams are continually altering their course within certain limits, as the current undermines the banks, or a fallen tree dams the stream, and it straightway finds a new channel for itself.

In many ways timing is a more convenient method of determining distances than pacing, as it is not so tedious, and there is less danger of error. It is so easy to lose count in pacing, and especially when with a column on the line of march, or with coolies on a narrow path, where slight checks are frequent. It is better then to measure the distances by timing, and the rate of the column will be a better one to take than one's own. Timing requires a good deal of attention, but between the observations of the watch, one is free to note objects on the line of march, and to watch the topography, whereas pacing demands one's whole attention throughout.

Clinometers and levels.

Various forms of clinometers and levels have been used for laying out approximate contours to guide the eye when sketching. In the Survey of India Department, a convenient portable form of clinometer is used of which a sketch and description are here given (*vide* fig. 66). By means of this instrument, the surveyor can measure the relative height of any object in view with regard to his own position, with the aid of only a simple calculation done on the spot. He can either deduce the height of his own position by observations to a fixed point of which the height is known, or to two or three such points for greater accuracy, or if the height of his position is known, he can obtain the height of any other point in view within reasonable distance. It is necessary of course to know the distance of the point observed, and this (which should not exceed three or four miles) may be measured off from the plane-table. To obtain the difference of height between his own position and any other object in view, the observer looks through the hole in the sight vane, after levelling the clinometer, and notes what figure on the tangent scale is cut by the ray to the object. This figure, multiplied by the distance in feet, gives the difference of height in feet between the observer and the object.

Having found the height of his position, the surveyor proceeds with the clinometer to determine the heights of a number of points in his vicinity, such as, the junctions of streams, prominent knolls, points on spurs where there is a marked change in slope, etc., entering their heights at once on the board. If the slope on which he stands is fairly uniform he can then find the position of points on the contours above and below him by observing the slope of the hill with his clinometer, and obtaining the horizontal equivalent for the difference in height between his position and that of the contour from the height indicator. The

distances of the contours next above and next below the observer's position being thus determined in several directions, the contours can be easily traced between the points so found, and the operation is then repeated at subsequent fixings, or if desired, the surveyor can set up his plane-table on the contour by finding his

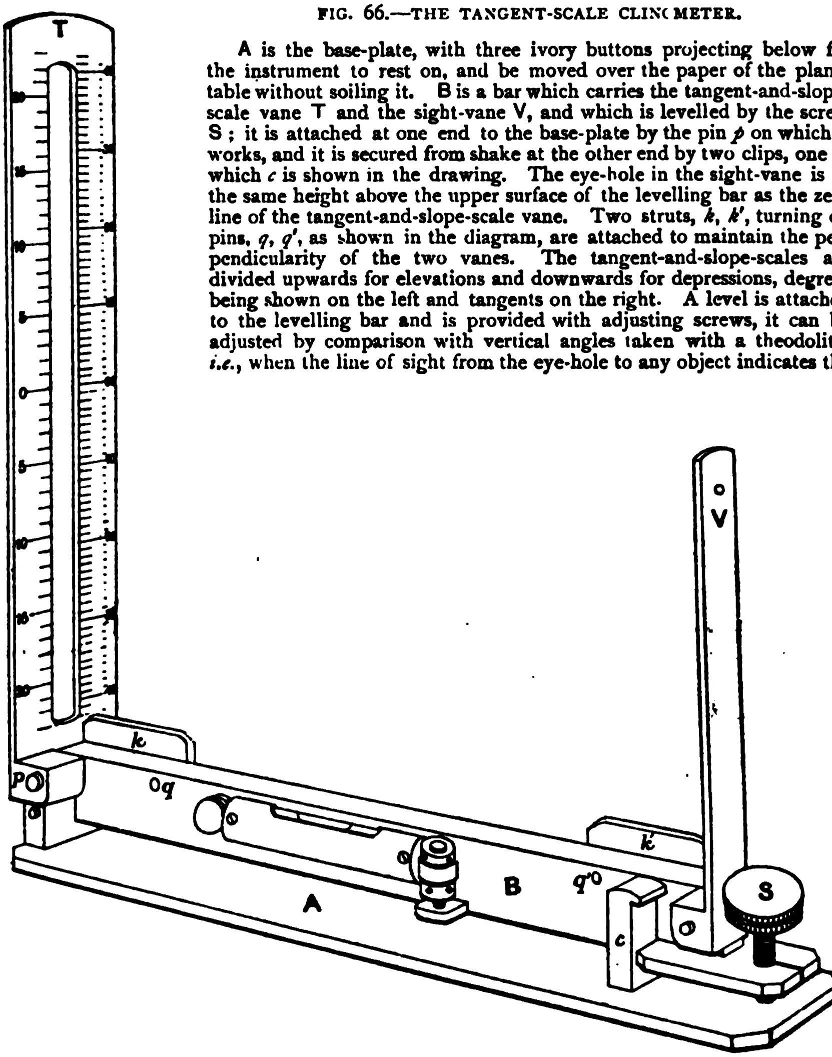


FIG. 66.—THE TANGENT-SCALE CLINOMETER.

A is the base-plate, with three ivory buttons projecting below for the instrument to rest on, and be moved over the paper of the plane-table without soiling it. B is a bar which carries the tangent-and-slope-scale vane T and the sight-vane V, and which is levelled by the screw S; it is attached at one end to the base-plate by the pin *p* on which it works, and it is secured from shake at the other end by two clips, one of which *c* is shown in the drawing. The eye-hole in the sight-vane is at the same height above the upper surface of the levelling bar as the zero line of the tangent-and-slope-scale vane. Two struts, *k*, *k'*, turning on pins, *q*, *q'*, as shown in the diagram, are attached to maintain the perpendicularity of the two vanes. The tangent-and-slope-scales are divided upwards for elevations and downwards for depressions, degrees being shown on the left and tangents on the right. A level is attached to the levelling bar and is provided with adjusting screws, it can be adjusted by comparison with vertical angles taken with a theodolite, *i.e.*, when the line of sight from the eye-hole to any object indicates the angle shown by a theodolite reading of the same object, the bubble should be in the centre of the level. The levels have been adjusted before being sent out, and the adjusting screws of the levels should not be touched except by a responsible officer.

Note.—Before observing, the struts *k k'* should invariably be turned up to avoid error due to want of perpendicularity, and care should be taken to turn them back again before attempting to shut up the instrument, otherwise the vanes are liable to be bent. This instrument is graduated to a radius of 8 inches, being the distance from the eye-hole V to the zero of the graduated arc.

position at one of the points on it, and using the clinometer as a level, lay out the contour in the usual way. Intermediate contours are then interpolated by eye with the aid of the heights already fixed at sufficiently close intervals to show all important features. Contours at 25-feet intervals are found suitable on surveys of 4 inches = 1 mile, and 50 feet on 2-inch surveys.

In surveys on smaller scales, it is impossible to bring out the hill features in satisfactory relief by contour lines at uniform vertical intervals. The relative steepness of slopes can be more effectively shown by judicious freehand, horizontal or vertical form lines, or hâchuring. Numerous heights should be fixed by the clinometer to supplement those given by the triangulation, and these are especially necessary at obligatory points required for engineering works, such as junctions of rivers, bridges, passes in ranges, etc.

Inking in work. It is difficult to lay down any rule for inking up a plane-table sheet. Some men can give an excellent idea of the country by brush work, and when done well this is effective, and besides names can be written over it without interfering with topographical detail, which is a decided advantage. As brush work cannot be done in the open, it is necessary to draw in all the features carefully in pencil on the spot with 'form lines,' and on the first opportunity, cover them with brush work. Some men can only indicate features by 'form lines' or hâchures, but whatever method is adopted all inking up should be done as soon as possible, either every night, or at each halt, while the topography is still fresh in the mind. Where villages are numerous, it is necessary to show them by numbers only, or all under features will be obliterated by the names. The spelling of names in a foreign country or in one which has no written character, must be as accurate as possible, and on some authorised system to ensure uniformity.

In parts of a district which are only *approximately* surveyed, all streams should be shown by dotted lines and the features so indicated as to distinguish them from accurate survey work, and to leave no doubt as to its inexact character. It may be convenient to add to the plane-table survey, work of a less exact nature than the surveyor's (such as rough native maps, routes performed by untrained or little trained assistants), and this should also be distinctly shown to be approximate, or unreliable. In the early days of our occupation of Burma, several sepoy surveyors and others, surveyed routes lying outside the writer's work, but as they generally started from some of his fixed points or known places and returned to others, he was able to adjust them to a certain extent, and they appeared in dotted lines on the first maps of the country. They were very useful, though the officers using the maps knew that they were not absolutely accurate. Where such routes are compiled on the plane-table sheet, notes should always be made to show whose work it is, and how it has been performed, as a guide to the value and reliability attaching to it.

CHAPTER XII.

SKETCHING, OR RAPID SURVEY WORK.

General Remarks.

THE methods of conducting the field work of surveys hitherto described, have been based on the assumptions, (1) that considerable accuracy is aimed at, (2) that ample time for the work is available, and (3) that a staff of trained assistants is to hand.

It will sometimes be found necessary to dispense with some or all of the above conditions, and to produce, in a very limited space of time, and without any skilled assistance, sketch maps of areas, or of linear work (such as projected roads, railroads, etc.), with a limited degree of accuracy, but nevertheless, which shall not fall short of what may be expected in small scale sketches.

The term 'sketch' is used here advisedly, since it is absolutely necessary that such hasty, and moderately accurate work should be qualified by the use of a 'heading' other than that of a 'survey' or 'plan.'

As with military reconnaissances, which vary in degree of accuracy from the roughest sketch to a near approach to a regular survey, so with work of the nature now referred to, the method to be adopted must depend on the object with which the sketch is made, the time and party available, and the amount of accuracy of detail and of finish to be aimed at.

Degree of Accuracy Discussed.

The degree of accuracy to be expected with a chain has been stated to be $\frac{1}{1000}$ over fairly level ground, or $\frac{1}{500}$ or more over rough and hilly ground. For rapid work this method of measuring is too slow and the use of a subtense compass or theodolite, a perambulator, or even pacing or estimation of distance by time, must be resorted to. The error of good pacing amounts to at least $\pm$ 1 per cent.

For observation of angular measurements, when sketching, a prismatic compass or magnetic needle used with a light plane-table, has to be relied on, when time will not admit of the use of a small theodolite (say 4 inch).

Let us now examine the relations which should subsist between the degree of accuracy of the instruments used, and the length of the sides measured, so that the unavoidable error may be unplottable, to a given scale.

In the following investigations a length of $\frac{1}{100}$ of an inch is assumed as just plottable on a plan.

(1) With a chain, supposing the limit of accuracy to be $\frac{1}{1000}$, and the scale

6 in. = 1 mile, or R.F.* = $\frac{1}{10560}$. Now, to this scale the length of the admissible error in inches will be $\frac{l}{1000 \times 10560}$,

Therefore

$$\frac{l}{1000 \times 10560} \text{ must not exceed } \frac{1}{100} \text{ of an inch,}$$

$$\text{or} \quad l \quad \text{,,} \quad \text{,,} \quad 100 \times 1056 = 105{,}600 \text{ inches}$$
$$= 1 \cdot 534 \text{ miles.}$$

(2) With pacing, assuming $\frac{1}{100}$ as limit of accuracy, and the scale 1 in. = 1 mile, or R.F. = $\frac{1}{63360}$.

$$\text{Here, } \frac{l}{100 \times 63360} \text{ must not exceed } \frac{1}{100} \text{ of an inch,}$$

$$l \quad \text{,,} \quad \text{,,} \quad 63{,}360 \text{ inches} = 1 \text{ mile.}$$

(3) With a prismatic compass, assuming that an angle can be read to within 30′ or $\frac{1}{2}°$ of the truth.

The angle which subtends an arc equal to the radius of a circle, being $57°\cdot3 = 206265''$ we have the arc subtended by 30′ to a radius l, equal to

$$\frac{l \times 30 \times 60}{206265}, \text{ which must not exceed } \frac{63360}{100} \text{ for a 1 inch scale,}$$

$$\text{or} \quad l \quad \text{,,} \quad \text{,,} \quad = \frac{206265 \times 63360}{30 \times 60 \times 100} \text{ inches.}$$
$$= 2076 \text{ yards or } 1\cdot15 \text{ miles, nearly.}$$

(4) With a pocket sextant reading to 1′, and to 6 inch scale, l = 6 miles.

(5) With a theodolite reading to 30″ and to 6 inch scale, l = 12 miles.

Generally, if θ = the least readable angle in seconds

$\frac{1}{a}$ = the least length which can be plotted in inches

l = the length of side required in feet

$\frac{1}{f}$ = R.F. of scale

then,
$$l = \frac{206{,}265 \times f}{\theta \times a \times 12}$$

The above estimates are a guide to the considerations which should govern the selection of instruments to be employed, and the scale to be adopted.

Levels. On sketches executed under the conditions above referred to, contour lines and heights above M.S. level, must be approximately determined by the use of a pocket aneroid barometer, or a hand levelling instrument, such as a protractor or quadrant with a weight suspended by a string, or one of the clinometers such as 'Abney's level' described in pp. 82–86 of Part I.

* R.F. for Representative Fraction.

Pocket Aneroid.

The pocket aneroid about $2\frac{1}{4}$ to $2\frac{1}{2}$ inches in diameter is found to be as reliable, for the determination of heights, as aneroids of much larger size. It is usually provided with a movable rim, indicating heights in feet, and by this means the scale can be set at any time to read a known height. The graduations read to 20 feet, and intermediate heights must be estimated by eye.

Sketches of Areas with Compass and Pacing.

Whenever pacing is used, each individual must first ascertain the length of his natural pace. A man of nearly 6 feet high can usually train himself to pace yards, when walking at 4 miles an hour, whilst shorter men must regulate their pace so that from 5 per cent. or 10 per cent. must always be *deducted*. In other cases it will be necessary to construct a 'comparative scale' of yards to suit the excess in the length of each 100 yards as paced.

Before taking a prismatic compass into use, its 'error' should always be determined, and this is especially necessary when a 'card compass' and not a 'metallic rim compass' is used, as the card zero is often inaccurately aligned with the magnetic axis of the needle. Again, in many localities the needle is affected to a more or less degree by local attractions such as large hills, presence of ironstone, etc.

When making the sketch of an area, it is best to first cover it with triangulated points, after selecting and carefully pacing a base line, and then taking rounds of angles with the prismatic compass, as when making a regular survey. In this case, however, the angles are generally plotted on the spot, and to enable this to be done, a sketching case or board of suitable and handy dimensions is used. Angles are plotted with a protractor, which is laid with one edge along or parallel to N. and S. lines or at right angles to E. and W. previously ruled at unequal distances on the paper. For areas, a larger board or case is required than for linear work, such as roads, etc.

Having fixed a number of points by a rough triangulation, further details are filled in by traversing in the usual manner, always closing when possible on the previously fixed points.

In wooded or flat districts triangulation is often impossible, and then traversing alone must be relied on, but in this case the sketch of the area should be built up, as it were, piece by piece, from a part to the whole, and whenever possible forward points should be fixed and closed on. When traversing, bearings should be taken to all prominent objects from time to time, using the intermediate traversed distance as a 'broken base.'

Finishing up Sketches.

When finishing up sketches the following points should be attended to.

(*a*) Print (in block printing preferable) a heading stating the name of the district, or describing the object of linear work.

(*b*) Draw a North point showing the declination of the needle by both 'true' and 'magnetic' north lines, a star being affixed to the former, and a fleur-de-lys to the latter.

(*c*) Arrange the work to suit the paper, but with ***approximate*** north side uppermost.

(*d*) Sketch in contour lines in red, blue, or chain dotted lines, for all scales of 2 inches = 1 mile, or larger.

(For smaller scales, hill features can only be shown by vertical or horizontal hâchuring, or pencil or colour shading.)

(*e*) State where all roads come 'from' on the left, and go 'to' on the right. This applies to cross-roads, railroads, etc., in linear work.

(*f*) Draw a scale of yards, and state the 'representative fraction.'

(*g*) The names of all towns and villages to be given in 'block printing.'

(*h*) Names of all rivers or streams to be given, with an 'arrow' showing the direction in which each water channel flows.

Linear Sketches.

For linear work, traversing can alone be resorted to, and pacing may have to give place to estimation of distances by time. Any conspicuous points, as far ahead as possible, should always be carefully observed to, as such bearings often serve as a check on the general direction of the work.

Finishing up Linear Sketches.

In finishing up linear sketches, the following points should be attended to.

(*a*) Use a scale of from ½ mile = 1 inch, to 4 miles = 1 inch, according to circumstances.

(*b*) Lateral communications should be shown and followed up for ¼ of a mile or so.

(*c*) Features of the ground on either side, to as great a distance as is visible, should be shown on either side of the sketch.

(*d*) All rivers, streams, bridges, culverts, telegraph lines, etc., should be shown.

General Principles affecting Sketches by more accurate Measurements.

The general principles which affect the methods of procedure, when more accurate means of determining angular and linear measurements are available, are the same as those above described when using a prismatic compass and pacing. The only difference lies in the time taken over the work, and the increased accuracy obtained when such instruments as a light theodolite, or a small plane-table are used, or measurements are made with a perambulator or a subtense instrument. If time admits, and the ground is not too much cut up, nothing approaches the accuracy in linear measurements made, even at the run, with a steel band (say 100 feet). The writer has found such measurements, in conjunction with the use of a light plane-table, work admirably. Two natives manipulated the tape, but it was found always necessary to await the change of arrows from 'follower' to 'leader' and to take the odd measurement to each new position of the plane-table.

Light Sketching Board.

Before leaving this subject, it will be well to describe the use of a light field sketching board, similar to the 'cavalry sketching case' at present in use in the army for mounted reconnaissance work.

To a $\frac{1}{4}$-inch light wooden board 7 inches long and 6 inches wide, are fastened two strips of copper, one on each side, and protruding at either end 1 inch beyond the board. In these ends, bulged holes are made, which carry the ends of two rollers, which are formed with a removable segmental strip which holds the paper. The paper, cut in long strips as required, is rolled on either of the rollers, the other end being passed over the board and clipped into the other roller.

Outside one of the copper strips a wooden piece is attached, of sufficient width to carry the small magnetic compass which is fixed in it.

To the centre of the back of the board, a leather wrist strap is fixed by a copper nail with a flat head, round which the strap can revolve.

A 1-inch flat ruler from 8 to 10 inches long is used with the board.

The glass cover of the compass is attached to a movable rim carrying a pointer, which is set, and used as a meridian mark.

The board thus mounted and ready for use weighs 1 lb. It is strongly built, and stands a good deal of knocking about.

A couple of strong elastic bands are found useful, and hold the ruler steady when slipped under them.

The method of procedure when using this instrument, is identical with that with a light plane-table, since the compass provides the means of orienting or holding the board at all times in the same meridian, and the ruler is used precisely as the alidade. With practice, the use of the ruler (which can have a scale on either edge) can be dispensed with, lines in any direction being drawn freehand.

If the ruler be not used, a cardboard scale 2 × 1 inches is necessary, and can be suspended by a string from the board.

Comparative scales of a horse's walk, trot, or canter, or time scales of yards, can be used conveniently with this instrument, or of ordinary pacing, when used dismounted.

Photographic Surveying.

General Remarks.

Since the year 1849, attempts have been made to use photography for mapping purposes, and in recent years many papers have been written on this subject, including a practical treatise by Mr. E. Deville, Surveyor-General of Canada, and published in 1895 by the Government printing bureau at Ottawa.

In the north-west of Canada many thousands of miles in area have been surveyed by metro-photographic methods with great success.

Owing to the great difficulty in obtaining good photographs, in this country, (showing distant details clearly), where the atmosphere is seldom clear for long

together, this system has not found favour, but in our Colonies such difficulties do not exist, and it is therefore deemed desirable to describe how such surveys are made.

The camera for taking the views should possess the following qualities, etc., viz.

1. A good lens with an aperture of 45°.

2 A firmly constructed body (not of collapsing pattern) to take flat glass plates.

3. Levelling screws, and levels to set the optical axis truly horizontal, and the picture plane and vertical rotating axis truly vertical.

4. Points, or a hair stretched, to mark on the negative the trace of the horizontal plane, and also the principal plane of the perspective view.

5. Some kind of sighting arrangement with which to direct the optical axis of the lens on objects.

6. The camera should be mounted in a vertical rotating axis, and have a graduated horizontal limb, on which to read the angle through which the instrument is rotated.

7. Clamps, verniers, microscopes, and tangent screws, etc., as for a theodolite.

Cameras made of metal are superior for this purpose to those made of wood, as there is then no fear of their losing shape or dimensions.

Method of Conducting a Photographic Survey. Field Work.

The following is a description of the method of conducting a photographic survey.

If the country to be mapped has been triangulated, first obtain a plan showing the trigonometrical points, and if no such triangulation has been made, the surveyor must first take a theodolite and make one for himself. Even if not convenient as camera stations, the triangulation points so determined, will serve as a means of fixing the exact position of the camera stations, as well as the orientation of the camera, when taking the views.

The aim in all cases is to obtain clear views of every part of the district to be mapped, from *at least* two stations, the ordinary requirements of any triangulation as to well-conditioned intersections, being observed, since fixings by acute or obtuse rays are obviously bad. A kind of scheme should be evolved for the general location of camera stations in the district to be mapped. The topographer in the field must bear in mind that the plotter in the office can only fix the positions of points that are visible in at least two photographs, and it will be often difficult to realise this necessary condition on account of intervening ground, such as spurs, knolls, etc. The number of views required at each station depends on the angle of view of the lens employed (i.e. on the ratio of the focal length of the lens to the size of the plate used).

Office Work.

The work to be done in the office consists in the preparation of the photographs from the undeveloped plates exposed in the field, and in plotting the topography with the aid of these photographs. Having carefully developed the negatives and secured good prints, the latter may be handed over to the topographer. It is possible to plot direct from photographs with only a 6-inch distance line, if very great care is taken, but as a rule

it is better not to plot from short-focus pictures, and as an alternative it is better to have enlargements made and to plot from them. By the use of enlargements, the traces of the picture planes may be arranged to fall outside the area being plotted, which is obviously a great convenience. There is a limit, however, to the size of an enlargement for convenient manipulation, and this may be fixed at about 3 feet. It is also convenient, though not necessary, that all the pictures used from a pair of stations for intersections, should be enlarged to the same scale.

The exact positions of the photograph stations having been plotted on the plan, the topographer selects a suitable pair of pictures, containing in each a convenient number of salient points to plot. He next makes fine red ink dots on corresponding points in each picture, writing a small number against each for identification. This work can really be best done by the field topographer, whose memory will aid him in identifying points in each picture. Unless very well defined objects, such as churches, detached houses, conspicuous monuments, trees struck by lightning, etc., can be found in each picture, the writer has found considerable difficulty in making the necessary identifications, and especially if the stations have little relative command over the area photographed. Mr. Deville, with his Canadian experience, however, states that this is not so, but a mistaken idea, since there is no difficulty whatever in identifying any number of points on moderately good photographs.

Having then marked points on each photograph, the next thing is to lay off from the 'stations' the 'distance lines.' These should be plotted with a good protractor, and great care must be taken to see that they are correctly 'oriented.' This must be ensured by careful observations taken in the field to known points (triangulated), for 'compass' bearings for this purpose cannot obviously produce exact work. The actual, or enlarged 'focal length' for the picture, is next laid off from each station along the 'distance line,' and through this point a perpendicular to the 'distance line' is drawn, this forming the 'trace' of the 'picture plane' of the selected view.

The plotter next takes two narrow strips of paper, as long as the picture (enlargement) is broad, and rules a fine line across the middle of each. On one of these bands he then marks off the exact distance of each dot on one of the views, right and left of the middle line, noting the numbers near each dot. These distances are then transferred to the trace of the picture plane. This process having been carried out with the view from another station, pins are driven at each station point, and two long fine threads, with loops at one end and a piece of elastic and a weight at the other, are attached to each station point. By placing each thread successively over dots with the same number in each trace, the intersections indicate the precise positions of each point, successively, as indicated by each dot. At this stage the topographer can note if the intersections in each case are 'well-conditioned.' The process above described can be repeated indefinitely with other photographs, from the same or other stations, and outlines between the points so plotted, can be filled in as the work progresses.

The above method is that which has been adopted in Canada and elsewhere, for plotting ground plans, and experience has shown that with reasonable care satisfactory results are attainable.

Contouring. Assuming that the camera has in every case been carefully levelled, so that the optical axis of the lens lies in a horizontal plane, it is evident that the horizon line on the picture will pass through all points at the same level as the station from which the view was taken. It is evident then that a line joining all the points plotted in plan which lie on this line, will be a contour line at a known level (for it is assumed that the heights of all the camera stations have been either calculated from vertical angles, or observed with a barometer). As stations will vary in height above sea-level in hilly country, a number of contour lines at uncertain intervals will thus be readily traced. If special contours be required, then the surveyor in the field should select some stations at the particular levels at which such contours are required to be traced.

Triangulation not absolutely necessary. Though desirable, and affording great facility in conducting the field work of a photographic survey, a triangulation of the area is not a 'sine quâ non.' Station points can be fixed as the work proceeds, from the photographs alone, but in this case a base line whose azimuth has been determined, must be measured between two stations from which to start the work. Later on, stations may sometimes be fixed by the station pointer from the positions of three points already mapped.

Suitable Country. It is clear that this method of surveying would be quite useless in flat districts or in densely wooded country, or for tracing the course of roads or streams through such districts. In well defined and hilly country, and to survey inaccessible peaks, etc., it has every chance of success. Though its cost in such cases or over suitable areas, may be only one-third of that by the ordinary methods, still we have no proof before us of its comparative accuracy, and we are therefore inclined to regard it more as a very useful adjunct to ordinary methods, in difficult situations, than as a method which could be universally adopted.

Mr. Bennett Brough has stated that this method is of special value to the mining engineer, whose work may lie in unhealthy malarial districts, and also that it is well adapted to surveys of mountainous districts and difficult country, for laying out mine railways or aërial wire rope-ways. Regions, such as ground above mines, where subsidence has set in, and which cannot be safely traversed, or areas to which access is forbidden by the owners, can be surveyed in this way.

Klondyke Survey. The map of the Klondyke gold-fields was recently plotted from sixteen photographs (taken under very adverse conditions), from three points at altitudes of 2870, 3700, and 3450 feet respectively, as determined by aneroid barometric observations.

Photo-Theodolite of Mr. Bridges Lees. The photo-theodolite as designed by Mr. Bridges Lees is reported as being a very ingenious and suitable instrument with which to make such surveys.

CHAPTER XIII.

BAROMETRIC AND HYPSOMETRIC DETERMINATIONS OF HEIGHTS ABOVE SEA LEVEL.

As referred to at the foot of page 66 of Part I. of this treatise, heights can be calculated from observations made with a barometer or a hypsometer, and a short description of the method of procedure, when either of these instruments is used, is here given.

Barometric Observations for Height.

The ever decreasing pressure of the atmosphere as increased heights above sea level are attained, suggests an obvious method of determining differences of altitude, by means of barometric observations. Were the atmosphere a homogeneous envelope, so that the pressures alluded to, decreased uniformly in arithmetrical progression, the necessary calculations would present no difficulty, but such is not the case, for the density of the air decreases in the ratio of a geometrical progression. If homogeneous, the envelope of air would be but five miles thick.

With whatever instrument the air pressure or density at various altitudes has been determined, whether with a mercurial barometer, an aneroid, or by the temperature at which water boils with the hypsometer, the same calculations have to be made in order to work out the relative heights at which the various observations were taken.

Excepting when used as described in Chapter XII. on sketching, or for determining differences of level at moderate intervals and within limited intervals of time, more or less synchronous observations of the air's density must be made at the stations whose difference in altitude it is desired to determine. This is obviously necessary, since the density and pressure of the air are materially affected by changes in temperature, and such changes will generally take place if observations are made at considerable differences in altitude above sea-level, or at several hours' interval of time.

Observations should be taken in as settled weather as possible, and if near a fixed meteorological station, at the same hours of the day at which observations are taken at such a station. As many observations as possible should be taken for each determination. Though it is impossible in the present state of physical science to effect any high refinement in the formula for computing barometric heights, still, fair results can be attained if the limits of error due to the instrument and methods of observation are carefully weighed, and attention paid to the hour of day (for the diurnal wave variations) and the month of the year (for monthly mean variations), as also to the degree of unsettledness of the weather at the time

the observations were taken, and effect given to the above in the calculation of the results.

From inattention to these simple considerations many important heights as published, are really most erroneous, and require revision.

The Mercurial Mountain Barometer. This portable barometer is constructed with a thumb-screw at its lower extremity, by turning which the mercury in the reservoir is confined within a limited space for transport purposes. In such cases, this end should be carried uppermost.

The most accurate results possible are to be obtained with this barometer, but it requires more careful handling than the aneroid. Observations must be corrected for the readings of both the 'attached' and 'detached' (air) thermometers.

The Aneroid Barometer. This well known and useful instrument is made of various sizes, and is compensated for internal effects of changes of temperature, but from this fact it does not follow that the calculations must not include corrections for the air temperature. This instrument must be frequently compared with a mercurial barometer, and even then none can lay any claim to the exactness of the latter. The mechanism is liable to get out of order, and they are slow in 'taking up' the effects of changes of pressure.

Its portability and handiness are a great recommendation, no doubt, and in many cases (especially if several be used as a check on each other), good results may be obtained.

The Hypsometer. The hypsometer is an instrument for determining the temperature of the boiling-point of water, from which barometric pressures can be directly deduced, and used as above indicated to calculate heights above sea-level. It consists of a cylindrical reservoir with a telescopic funnel, from the top of which when drawn out and in use, the steam escapes through a small hole, and through which the stem of a thermometer (whose bulb is immersed in the water in the reservoir) protrudes, and can be read. The reservoir is heated by a spirit lamp.

Formulæ for Calculating Heights. Various formulæ have been devised and tables calculated for computing the difference in height between two stations by means of the observed heights of the barometer, of which the following are examples.

(1) By Laplace—

$$\text{Difference in height in feet} = 60345{\cdot}51$$
$$\times \{1 + {\cdot}00111111\ (\tau + \tau' - 64^\circ)\}$$
$$\times \log \left\{\frac{\beta}{\beta'} \times \frac{1}{1 + {\cdot}0001\ (t - t')}\right\}$$
$$\times \{1 + {\cdot}002695 \cos 2\phi\}$$

Where $\beta\ \beta'$ = barometric readings

$t\ t'$ = readings of 'attached' thermometers

$\tau\ \tau'$ = readings of air temperatures, or 'detached' thermometers

ϕ = latitude (mean of stations).

The *unaccentuated* figures are the readings at the *lower* of the two stations.

(2) By Poisson—

$$\text{H in yards} = \text{A}\left\{\log \beta - \log\left[\beta' \times \left(1 + \frac{t - t'}{9990}\right)\right]\right\}$$

Where $\beta\,\beta'$ = barometric readings (β at lower station)

$$\text{A} = \frac{20053 \cdot 95}{1 - \cdot 002588 \cos\phi}\left(1 + \frac{\tau + \tau' - 64^\circ}{900}\right)$$

$\tau\,\tau'$ = readings of air temperatures, or 'detached' thermo meters

$t\,t'$ = readings of 'attached' thermometers

ϕ = latitude (mean of stations)

(3) Given by Colonel Mackesy in his 'Tables of Barometric Heights' for use with the compensated aneroid barometer.

$$\left.\begin{array}{c}\text{Difference in height in feet}\\ \text{between two stations}\end{array}\right\} = 67 \cdot 05\,(836 + t + t') \times (\log h - \log h')$$

Where $t\,t'$ = air temperatures at the *lower* and *upper* stations respectively

$h\,h'$ = barometric readings at *lower* and *upper* stations respectively

This formula can be used with the mercurial barometer if the readings are first corrected for capillarity and reduced to 32° F.

(4) To connect observations of the boiling-point as indicated by the hypsometer, with corresponding barometric readings.

$$h = 29 \cdot 92 - \cdot 59\,t$$

Where h = corresponding height of mercurial column in inches

t° = degrees F., by which the boiling point, as read on the hypsometer, is less than 212° F.

The air temperature must be taken at each observation for use in the formula selected for calculating altitudes by the observed (in this case calculated) heights of the barometric column.

For further information, *vide* Appendix I.

Note.—In both formulæ the factor involving τ, τ', the readings of the air thermometer or "detached" thermometer at upper and lower stations, corrects for the mean density of the intervening air stratum, which is *assumed* to be the arithmetical mean of the upper and lower air-temperatures. This assumption is not always correct. Hence barometric levels are always uncertain.

The factor containing t and t' corrects the barometric reading for the effect of temperature on the mercury and scale, and is therefore unnecessary with the aneroid and with the hypsometer.

CHAPTER XIV.

PROJECTIONS AND SUN-DIALS.

Preliminary Remarks.

PROJECTIONS are considered in this chapter in connection with the representations of portions of the 'earth's geodetic surface' on a 'plane surface.'

It is clear that all such representations must be more or less distorted, and it therefore behoves us to adopt that method which shall best fulfil the requirements of each special case, with the least possible error.

Projections Divided into Two Classes.

Projections may be divided into two classes, as follows—

(1) Those which are perspective representations.
(2) Those which are developments of areas preserving certain geometrical relations to the true figure.

Class (1) are used for maps of the world, hemispheres, or portions thereof.

Class (2) are used for the representation of smaller areas, in which a minimum distortion is sought.

To Class (1) belong, Orthographic, Stereographic, Globular, Sir H. James's, and Gnomonic projections.

To Class (2) belong, Rectangular, Trapezoidal, Rectangular Tangential, Colonel Blacker's, Conical, and Mercator's projections.

Principles Leading to Choice of Method to be Adopted.

When the survey is a continuous traverse of a road or railroads, *actual* lengths are wanted, sea-level dimension not being aimed at, and such a representation is required that a scale may be applied to any part, and read actual distances. This, also, without reference to the relative altitudes above sea-level, or the earth's curvature.

When, however, the survey extends over a large area, some method must be adopted which will admit of reduction to the sea-level and the curvature of the earth. The particular method, then, to be employed depends largely on the following considerations.

(1) The extent of the region to be mapped.
(2) The use to be made of the map or chart.
(3) The degree of accuracy to be aimed at.

Definitions.

Before investigating the merits of various projections in detail it is desirable to remind the student of the following definitions.

The Projection of an Area. The projection of an area consists in the representation on a 'plane surface' of various points and lines contained in it (and delineating its form), as they would appear to the eye if placed in a particular position above or below the said 'plane.'

Plane of Projection. The 'plane,' above referred to, on which the delineation is made, is called the 'plane of projection.'

Point of Sight. The 'point' where the eye is supposed to be situated, is called the 'point of sight.'

Centre and Axis. The point in the 'plane of projection' at which a 'perpendicular' thereto, dropped from the 'point of sight,' meets the 'plane,' is called its 'centre,' and this 'perpendicular' is called its 'axis.'

Original. Any point, line, or other object, to be projected from its position on the 'spherical surface' is called the 'original.'

Projecting Line. The 'straight line' drawn from the 'point of sight' to any 'original' is called a 'projecting line.'

Projections of Class (1) will now be shortly described.

Orthographic Projection. In this projection the 'point of sight' is at an infinite distance above the hemisphere on which the given area, being the 'original,' is situated, and the latter is projected on the diametral plane, by lines falling perpendicularly on the same through points, etc. in the given area.

It is obvious that only central portions are represented in true form, so that this projection is only suited to very small areas, and is little used.

Stereographic Projection. In this projection one of the poles of any great circle of the sphere is selected as the 'point of sight,' and the plane of this great circle as the 'plane of projection.'

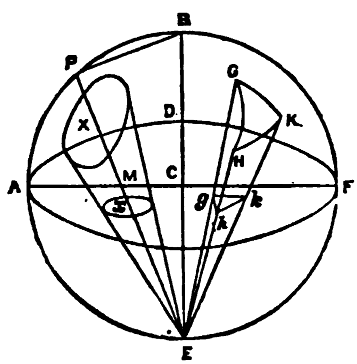


FIG. 67.

The projection is to a great extent free from the defects inherent in the 'orthographic' method. This projection is fully discussed in 'Chambers' Mathematics,' but the following short description is here given.

In fig. 67, suppose the eye to be placed at E, one extremity of the diameter E C B, of the sphere, and to view the concave surface of the sphere, every point of which, as P, is referred to the diametral plane A D F, perpendicular to E B, by the visual line P M E. The stereographic projection of a sphere, then, is a true perspective representation of its concavity on a diametral plane, and as such, it possesses some singular geometrical properties, of which the following are two of the principal.

Firstly, all circles on the sphere are represented by circles in the projection. Thus the circle X is projected into *x*. Great circles passing through the

vertex B are projected into straight lines traversing the centre C, thus B P A is projected into C A.

Secondly, every small triangle G H K, on the sphere, is represented by a *similar* triangle *g h k*, in the projection. This is a very valuable property, as it ensures a general similarity of appearance in the map, to the reality, in all its parts, and enables us to project at least a hemisphere in a single map, without any violent distortion of the configurations on the surface from their real forms. In orthographic projection, the borders of the hemisphere are unduly crowded together, in stereographic, their projected dimensions are, on the contrary, somewhat enlarged in receding from the centre.

The laws of this projection are as follows:—

(1) Every 'small' or 'great circle' whose plane passes through the 'point of sight' becomes a 'straight line,' *equal* arcs being *unequally* projected.

(2) The projection of a 'circle' parallel to the P. of P., will be a 'circle,' but of *greater* or *less* radius in proportion as the 'original' circle is *nearer to* or *further from* the 'point of sight' than the P. of P.

(3) The projection of a 'circle' which is oblique to the P. of P. will be a 'circle.'

(4) The projection of the arc of a 'great circle' passing through the 'point of sight' will be a 'straight line' equal to the tangent of half that arc.

(5) The 'angles' made by 'circles' intersecting each other on the P. of P. are equal to the 'angles' made by the 'originals' (circles).

This 'projection,' from the above properties, lends itself in a marked degree, to the construction of star-maps made to revolve under a transparent chart of altitude and azimuth circles.

Globular Projection. This method is not exactly speaking a projection. In it, the 'polar' and 'equatorial radii' are divided into the same number of parts, say nine, in which case the 'meridians' and 'parallels' would be at 10° intervals. The 'point of sight' is supposed to be situated at such a distance from the surface of the sphere, that equal arcs shall be represented by nearly equal portions in the plane of projection. The height of the 'point of sight' then = sin 45° when $r = 1$ or, say $\frac{3}{4}$ r.

Sir H. James's Projection. In this 'projection' the 'point of sight' is placed at a distance of *half* a radius of the globe *below* the spherical surface, the P. of P. being made to coincide with that of a small circle which is removed 23° 30′ from the equator on the side of, and towards the 'point of sight.'

Gnomonic Projection. Here the 'point of sight' is the centre of the sphere, and the 'planes of projection' are the sides of a circumscribing cube. The projection is suitable for star maps, but not for geographical purposes.

The following projections of Class (2), are more suitable than the above for mapping moderate-sized 'districts' or 'areas.'

Rectangular Projection. In this projection, the meridians are all drawn as straight parallel lines, and the parallels are also straight and at right-angles to the meridians. A central meridian, A B (fig. 68) is

drawn, and divided into minutes of latitude according to the value of these at that latitude, which may be obtained from tables for the purpose. Through these points of division draw the parallels of latitude, C D, etc., as straight lines perpendicular to the central meridian. On the central parallel, C D, lay off the minutes of longitude, according to their value for the given latitude, and through these points of division draw the other meridians parallel to the first.

The largest error here is in assuming the meridians to be parallel. For the latitude of 40°, two meridians a mile apart will converge at the rate of about a foot a mile. A knowledge of this fact will enable the draughtsman to decide when this method is sufficiently accurate for his purpose. Thus, for an area of 10 miles square, the distortion at the extreme corners in longitude, with reference to the centre of the map as an origin of co-ordinates, will be about 25 feet. At the equator this method is strictly correct.

In this kind of projection, whether plotted from polar or rectangular co-ordinates, or from latitudes and longitudes, all straight lines of the survey, whether determined by triangulation, or run out by a transit on the ground,

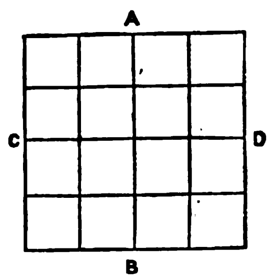


FIG. 68.

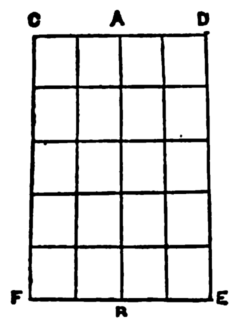


FIG. 69.

will be straight on the map, that is, the fore and back 'azimuths of a line' are the same, or in other words, a straight line on the map crosses all the meridians at a constant angle.

This is the method to use on field sheets, where the whole survey has been referred to a single meridian.

Trapezoidal Projection.

In trapezoidal projection (fig. 69) the meridians are made to converge properly, but both they and the parallels are straight lines. A central meridian is first drawn and graduated to degrees or minutes, and through these points parallels are drawn, as before. Two of these parallels are selected, one about one-fourth of the height of the map from the bottom, and the other the same distance from the top. These parallels are then subdivided, according to their respective latitudes, and through the corresponding points of division the remaining meridians are drawn as straight lines. The map is thus divided into a series of trapezoids. The parallels are only perpendicular to one of the meridians. The principal distortion comes from the parallels being drawn as straight lines, and amounts to about 32 feet in

10 miles in latitude 40°, and is nearly proportional to the square of the distance east or west from the central meridian.

The work should be plotted from computed latitudes and longitudes. This method is adapted to a scheme which has a system of triangulation for its basis, the geodetic position of the stations having been determined. These conditions would be fulfilled in a State topographical or geological survey, for the separate sheets, each sheet covering an area of not more than 25 miles square.

Rectangular Tangential Projection.

This projection was designed under the orders of the Director General of the Ordnance Survey, and is described in a paper published at the Ordnance Survey, entitled 'On the Rectangular Tangential Projection of the Sphere and Spheroid.' It is also used at the Intelligence Department and described in a memorandum published in 1890, headed 'Memo and Tables for the Projection of Maps for Military Purposes.' It is not considered necessary to describe this projection further here.

Col. Blacker's Projection for Small areas to Large Scales.

The above projection not being entirely satisfactory for the representation of small areas to large scales, Colonel Blacker invented the following convenient and accurate method, for plotting the 'parallels of latitude' and 'meridians' in such cases.

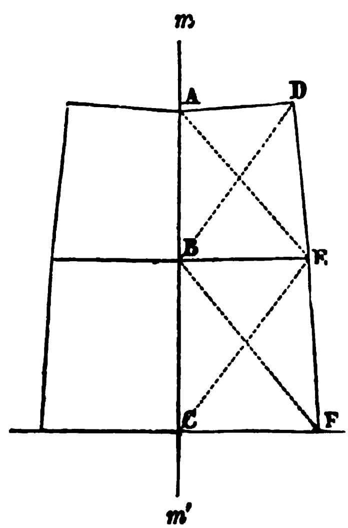


FIG. 70.

Along a central line, *m*, *m'* (fig. 70) representing the mean 'meridian of longitude' of the area, set off points A, B, C, etc., in which 'parallels of latitude' would cut this meridian (say ½° intervals). Quadrilateral figures are then built up on these meridional distances by means of the lengths of the diagonals taken from tables, the points D, E and F, being obtained by the intersections of arcs. The projection of the arcs of 'parallels' and 'meridians' are thus plotted on either side of the central meridian. Evidently as each new meridian is plotted, it serves as a base on which to build up further quadrilaterals, and so on.

For further information on this projection the student is referred to the 'Manual of Surveying for India.'

One of the great disadvantages of this procedure is that errors are *accumulative*, owing to the building-up system adopted.

Conical Projection.

In this 'projection' the 'point of sight' is the centre of the sphere, and the 'plane of projection' is a conical envelope whose surface is tangential to the 'middle latitude' of the area to be delineated. The envelope is ultimately divided and unrolled. If the equator be the middle latitude, then the conical envelope becomes a hollow cylinder.

This projection is well suited to the representation of small portions of the sphere, but if the map extend much on either side of the 'middle latitude,' the distortion becomes considerable.

Mercator's Projection. This projection is universally used in the construction of navigation charts. The 'meridians of longitude' are drawn as 'parallel straight lines' at their 'equatorial distance' apart, and 'parallels of latitude' are drawn at right-angles to these 'meridians' at distances augmented in the same ratio as the distances between the 'meridians of longitude' have been increased above their real values at each 'parallel of latitude.' The effect of this is, that the course of a ship sailing on a constant 'bearing' is plotted as a straight line, thus greatly simplifying the work of the navigator.

Problems which have generally to be solved for Mapping Purposes. Having indicated the various methods which may be employed when mapping surveys, it is now desirable to state and explain the problems which have most generally to be solved in such cases. They are as follows:—

(1) Given the 'lat.' and 'long.' of one station, together with the 'distance' and observed 'azimuth' to another, to compute the '*lat.*' *and* '*long.*' *of the latter*, together with *the reverse 'azimuth' to the first.*

(2) Given the 'lats.' and 'longs.' of two stations, to compute the '*distance*' *between them, and their mutual* '*azimuths.*'

(3) Given the 'lats.' of two stations, and the 'azimuth' of one from the other, to compute *their* '*distance*' *apart*, and their '*difference of longitude.*'

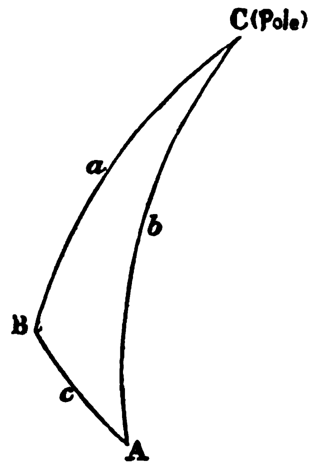


FIG. 71.

Problem (1). Treating the earth as a sphere, if A (fig. 71) is a point whose 'latitude' and 'longitude' are known, A B a given 'length' of which the 'azimuth' of B from A is known, then in the spherical triangle A B C (in which C is the pole), we know b (90° − latitude of A), c (distance A B), and the angle A.

Having reduced c from 'distance in feet' to 'seconds of arc,' by means of Table I., we can solve the triangle in the ordinary way by using the spherical formulæ:

$$\left.\begin{aligned} \tan \tfrac{1}{2}(B+C) &= \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)} \cot \frac{A}{2} \qquad \ldots \quad (a) \\ \tan \tfrac{1}{2}(B-C) &= \frac{\sin \frac{1}{2}(b-c)}{\sin \frac{1}{2}(b+c)} \cot \frac{A}{2} \qquad \ldots \quad (b) \\ \frac{\sin a}{\sin b} &= \frac{\sin A}{\sin B} \qquad \ldots \quad (c) \end{aligned}\right\}$$

From (a) and (b) we get—

B = reverse 'azimuth' of A from B.

and C = 'difference in longitude' between the two stations.

TABLE I.—TABLE FOR REDUCING 'LENGTH IN FEET' to 'SECONDS OF CONTAINED ARC.'

Lat.	Geodetic Distance in Feet.								
	10,000	20,000	30,000	40,000	50,000	60,000	70,000	80,000	90,000
°	″	″	″	″	″	″	″	″	″
0	98·57	197·13	295·70	394·27	492·84	591·40	689·97	788·54	887·10
2	·57	·13	·70	·27	·83	·40	·97	·53	·10
4	·57	·13	·70	·26	·83	·39	·96	·52	·09
6	·56	·13	·69	·25	·82	·38	·94	·51	·07
8	·56	·12	·68	·24	·80	·36	·92	·49	·04
10	·56	·11	·67	·23	·78	·34	·90	·46	·01
12	98·55	197·10	295·66	394·21	492·76	591·32	689·87	788·42	886·97
14	·55	·09	·64	·19	·74	·28	·83	·38	·93
16	·54	·08	·62	·17	·71	·25	·79	·33	·88
18	·54	·07	·60	·14	·68	·21	·75	·28	·82
20	·53	·06	·58	·11	·64	·17	·70	·22	·75
22	98·52	197·04	295·56	394·08	492·60	591·12	689·64	788·16	886·68
24	·51	·02	·54	·04	·56	·07	·58	·09	·61
26	·50	·01	·51	·01	·51	·02	·52	·02	·52
28	·49	196·99	·48	393·97	·47	590·96	·45	787·95	·44
30	·48	·97	·45	·93	·42	·90	·38	·87	·35
32	98·47	196·95	295·42	393·89	492·37	590·84	689·31	787·78	886·26
34	·46	·92	·39	·85	·31	·77	·24	·70	·16
36	·45	·90	·35	·81	·26	·71	·16	·61	·06
38	·44	·88	·32	·76	·20	·64	·08	·52	885·96
40	·43	·86	·29	·72	·14	·57	·00	·43	·86
42	98·42	196·83	295·25	393·67	492·09	590·50	688·92	787·34	885·75
44	·41	·81	·22	·62	·03	·43	·84	·24	·65
46	·39	·79	·18	·58	491·97	·36	·76	·15	·54
48	·38	·76	·15	·53	·91	·29	·68	·06	·44
50	·37	·74	·11	·48	·85	·22	·59	786·96	·34
52	98·36	196·72	295·08	393·44	491·79	590·15	688·51	786·86	885·23
54	·35	·70	·04	·39	·74	·09	·44	·78	·13
56	·34	·67	·01	·35	·68	·02	·36	·70	·03
58	·33	·65	294·98	·30	·63	589·96	·28	·61	884·94
60	·32	·63	·95	·26	·58	·90	·21	·53	·84

Computed from formula—

$$c'' = \frac{\text{length in feet} \times 180 \times 360}{\pi \nu}$$

Where ν = normal to the meridian terminated by the minor axis.

Calculated on the following constants—

Major axis = 41,852,696 feet.
Minor axis = 41,710,466 feet.

TABLE II.—VALUES OF QUANTITY * TO BE APPLIED FOR 'CORRECTION OF SPHERICALLY COMPUTED DIFFERENCE OF LATITUDE.' [*Vide* PROBLEMS (1), (2) AND (3).]

Lat.	Log.	Natural Number K.	Lat.	Log.	Natural Number K.
°			°		
0	·0029568	1·00683	30	·0022195	1·00512
1	9559	83	31	1744	02
2	9532	82	32	1285	1·00491
3	9487	81	33	0818	81
4	9425	80	34	0344	70
5	9344	78	35	·0019863	58
6	·0029246	1·00676	36	·0019375	1·00447
7	9130	73	37	8882	36
8	8997	70	38	8384	24
9	8847	66	39	7882	13
10	8679	63	40	7375	01
11	·0028495	1·00658	41	·0016866	1·00389
12	8294	54	42	6354	77
13	8077	49	43	5840	65
14	7843	43	44	5325	53
15	7594	37	45	4809	42
16	·0027329	1·00631	46	·0014293	1·00330
17	7048	25	47	3778	18
18	6753	18	48	3264	06
19	6444	11	49	2751	1·00294
20	6120	03	50	2241	82
21	·0025782	1·00595	51	·0011734	1·00271
22	5431	87	52	1231	59
23	5067	79	53	0732	47
24	4690	70	54	0238	36
25	4302	61	55	·0009750	25
26	·0023902	1·00552	56	·0009267	1·00214
27	3490	42	57	8792	03
28	3058	33	58	8323	1·00192
29	2636	23	59	7863	81
30	2195	12	60	7411	71

* This quantity is $\frac{\nu}{\rho}$ where ν = normal to meridian terminated by the minor axis, and ρ = radius of curvature to the meridian.

From (c) we get—

$$a = 90 - \text{latitude of B}$$

Now B and C are correct, but the latitude must be corrected for the false assumption that the earth is a sphere by applying the correction given in Table II.:—

i.e. log *true* diff. in lat. = log ($b - a$) + log from Table II.

Problem (2). In this case we are given the 'latitudes' and 'longitudes' of A and B, i.e. we know a, b, and C.

We require the 'distance' c, and the 'azimuths' A and B.

The formulæ will be similar to those given above—

$$\left.\begin{aligned} \tan \tfrac{1}{2}(A + B) &= \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2} \quad \ldots \quad (d) \\ \tan \tfrac{1}{2}(A - B) &= \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2} \quad \ldots \quad (e) \\ \frac{\sin c}{\sin b} &= \frac{\sin C}{\sin B} \quad \ldots \quad (f) \end{aligned}\right\}$$

Before using the *spherical* formulæ, the reverse of the operation for correcting the 'latitude'—explained above for Problem (1)—must be gone through.

i.e. log *spherical* diff. of lat. = log ($b - a$) − log from Table II.

Problem (3). Given the 'latitudes' of A and B and the 'azimuth' of B from A, required the 'difference of longitude,' and the 'distance' A B. In this case we know two sides a and b, and the angle opposite one of them A, but before solving the triangle by means of the ordinary spherical formulæ given below, a small correction (P) must be applied to the values of the 'co-latitudes,' as follows.

From Table II. take out the natural number 'K,' corresponding to the *mean* latitude of A B, then—

$P = \frac{d}{2} \frac{K - 1}{K}$, where d = 'difference in seconds' between 'latitudes' of A and B, and—

$$a = 90^\circ - \text{latitude of B} + P$$
$$b = 90^\circ - \text{latitude of A} - P$$

Formulæ:

$$\left.\begin{aligned} \tan \theta &= \tan b \cos A \\ \cos \theta' &= \frac{\cos a \cos \theta}{\cos b} \\ c &= \theta - \theta' \end{aligned}\right\} \text{To find } c \text{ or A B} \quad \ldots \quad (g)$$

$$\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c} \quad \text{This gives C, or 'diff. of longitude'} \quad \ldots \quad (h)$$

If it be required to find C independently of c, the following formulæ may be used:

$$\left.\begin{aligned} \tan \phi &= \frac{\cot A}{\cos b} \\ \cos \phi' &= \cos \phi \cot a \tan b \\ C &= \phi - \phi' \end{aligned}\right\} \quad \ldots \quad (j)$$

The 'difference of longitude' in seconds may be more *roughly* found from the formula—

$$\text{Difference of long. in secs.} = \frac{1}{K} \tan A \sec \lambda^{A} \times d \quad . \quad . \quad (k)$$

where λ_A = lat. of A

K and d as above.

The following examples illustrate Problems (1) and (3), and the necessary tables are attached.

Examples of Problem (1).

At station A, given,

$$\left.\begin{array}{lll} \text{Latitude} & = 12^\circ \text{ N.} & = 90^\circ - b \\ \text{Longitude} & = 3^\circ \text{ E.} & = C \\ \text{Distance A B} & = 15 \text{ miles} & = c \end{array}\right\} \quad . \quad . \quad .\text{(fig. 71)}$$

Azimuth of B from A = 135° (measured from S. by W.).

Required the lat. and long. of B, and azimuth of A from B.

It is first necessary to reduce c to 'distance in arc.'

This is done by the use of Table I. with arguments approximate mean latitude (12° 5') and distance in feet (79,200), thus:—

$$\begin{array}{rcl} 70{,}000 \text{ ft.} & = & 689''\cdot 87 \text{ at lat. } 12^\circ\ 5'' \\ 9000 \text{ ,,} & = & 88''\cdot 697 \\ 200 \text{ ,,} & = & \underline{1''\cdot 971} \\ & & \underline{780''\cdot 54} = 13'\ 0''\cdot 54 \end{array}$$

$$\therefore \quad \frac{b+c}{2} = 39^\circ\ 6'\ 30''\cdot 27 \qquad A = 45^\circ$$

$$\frac{b-c}{2} = 38^\circ\ 53'\ 29''\cdot 73 \qquad \frac{A}{2} = 22^\circ\ 30'$$

$$\begin{array}{rl|rl} \text{L} \cos \dfrac{b-c}{2} = & 9\cdot 8911668 & \text{L} \sin \dfrac{b-c}{2} = & 9\cdot 7978551 \\ \text{L} \cos \dfrac{b+c}{2} = & 9\cdot 8898359 & \text{L} \sin \dfrac{b+c}{2} = & 9\cdot 7998846 \\ & 0\cdot 0013309 & & \bar{1}\cdot 9979705 \\ \text{L} \cot \dfrac{A}{2} = & 10\cdot 3827757 & & 10\cdot 3827757 \\ \text{L} \tan \dfrac{B+C}{2} = & 10\cdot 3841066 & \text{L} \tan \dfrac{B-C}{2} = & 10\cdot 3807462 \end{array}$$

$$\frac{B+C}{2} = 67^\circ\ 33'\ 43''\cdot 24$$

$$\frac{B-C}{2} = 67^\circ\ 24'\ 10''\cdot 65$$

$$\begin{array}{l} \text{Sum} = B = 134^\circ\ 58'\ 1''\cdot 89 \\ \text{Diff.} = C = \qquad 9'\ 24''\cdot 59 \end{array}$$

$$\begin{array}{ll} \therefore \text{ Az. of A from B} & = 314^\circ\ 58'\ 1''\cdot 89 \\ \text{and Long. of B} & = 2^\circ\ 50'\ 35''\cdot 41 \text{ E.} \end{array}$$

$$\begin{aligned} L \sin A &= 9{\cdot}8494850 \\ L \sin b &= \underline{9{\cdot}9904044} \\ &\quad 19{\cdot}8398894 \\ L \sin B &= \underline{9{\cdot}8497335} \\ L \sin a &= 9{\cdot}9901559 \end{aligned}$$

$$\therefore \quad a = 77^\circ\ 50'\ 48''{\cdot}4$$

And approx. lat. of B $= 12^\circ\ \ 9'\ 11''{\cdot}6$

Correction of Latitude.

$$(b - a) = 9'\ 11''{\cdot}6 = 551''{\cdot}6$$

$$\begin{aligned} \log 551{\cdot}6 &= 2{\cdot}7516243 \\ (\text{for lat. } 12^\circ\ 4\tfrac{1}{2}')\ \log \text{Tab. II.} &= \underline{\ {\cdot}0028278} \\ &\quad 2{\cdot}7444521 \\ &= \log 555{\cdot}23 \end{aligned}$$

$$\therefore \quad \text{true diff. of lat.} = 9'\ 15''{\cdot}23$$

$$\therefore \quad \text{true lat. of B} = 12^\circ\ 9'\ 15''{\cdot}23 \text{ N.}$$

Example of Problem (3). At station A and B, given lat. of A $= 12^\circ$

„ „ B $= 12^\circ\ 9'\ 15''{\cdot}37$

Azimuth of B from A $= 45^\circ$

Required the 'diff. of long.' between A and B, and the 'distance' A B.

Now 'K' or Nat. No. for lat. $12^\circ\ 5'$ from Table II. $= 1{\cdot}00653$.

$$\therefore \quad P = \frac{9'\ 15''{\cdot}37}{2} \times \frac{{\cdot}00653}{1{\cdot}00653} = 1''{\cdot}8015$$

$$\therefore \quad a = 90^\circ - 12^\circ\ 9'\ 15''{\cdot}37 + 1{\cdot}8'' = 77^\circ\ 50'\ 46''{\cdot}43$$

and

$$b = 90^\circ - 12^\circ - 1''{\cdot}8 = 77^\circ\ 59'\ 58''{\cdot}2$$

Now

$$\begin{aligned} L \tan b &= \ {\cdot}6725069 \\ L \cos A &= \underline{9{\cdot}8494850} \\ L \tan \theta &= 10{\cdot}5219919 \end{aligned} \qquad \begin{aligned} L \cos a &= 9{\cdot}3233265 \\ L \cos \theta &= \underline{9{\cdot}4592214} \\ &\quad 18{\cdot}7825479 \\ L \cos b &= \underline{9{\cdot}3178967} \\ L \cos \theta' &= 9{\cdot}4646512 \end{aligned}$$

$$\left\{ \begin{aligned} \theta &= 73^\circ\ 16'\ 6''{\cdot}7 \\ \theta' &= \underline{73^\circ\ 3'\ 6''{\cdot}16} \end{aligned} \right.$$

$\therefore \quad c = 0^\circ\ 13'\ 0''{\cdot}54$ or 'distance A B' $= 79{,}200$ feet $= 15$ miles (from Table I.)

$$
\begin{array}{llll}
\text{L cot A} = & 10\cdot 0000000 & & \\
\text{L cos } b = & \underline{9\cdot 3178967} & \text{L tan } b = & \cdot 6725069 \\
\text{L tan } \phi = & \cdot 6821033 & \text{L cot } a = & 9\cdot 3331715 \\
\begin{cases} \phi = 78^\circ\, 15'\, 15''\cdot 83 \\ \phi' = 78^\circ\ \ 5'\, 54''\cdot 25 \end{cases} & & \begin{array}{l}\text{L cos } \phi = \\ \text{L cos } \phi' = \end{array} & \begin{array}{l}\underline{9\cdot 3087065} \\ 19\cdot 3143849\end{array}
\end{array}
$$

$$C = 0^\circ\ 9'\ 24''\cdot 58 = \text{Difference in longitude.}$$

Or, using the result obtained for c—

$$
\begin{aligned}
\text{L sin A} &= 9\cdot 8494850 \\
\text{L sin } c &= \underline{7\cdot 5779689} \\
& \quad 17\cdot 4274539 \\
\text{L sin } a &= \underline{9\cdot 9901550} \\
\text{L sin C} &= 7\cdot 4372989
\end{aligned}
$$

$\therefore\quad C = 0^\circ\ 9'\ 24''\cdot 58 =$ diff. in long. between A and B.

SUN-DIALS.

Definition of Sun-Dial. A '*sun-dial*' is a surface, generally a plane, on which a system of lines is drawn, in such a manner that the coincidence of the shadow of a *straight rod or edge* with any of them, points out the hour of the day in apparent time.

Stile or Gnomon. The 'straight rod or edge' is called the '*stile*,' or '*gnomon*' of the 'dial.'

Hour Lines. The lines are called '*hour lines*.'

Plate Stile. When the 'stile' is the edge of a plate, the latter is called a '*plate stile*.'

Sub-stile. The plane of the 'plate stile' is generally placed perpendicularly to the 'plane of the dial,' and its intersection with the 'plane of the dial' is called the '*sub-stile*.'

Elevation of the Stile. The inclination of the 'stile' to the 'plane of the dial'—that is, to the 'sub-stile'—is called the 'elevation of the stile.'

The 'stile' is always placed parallel to the 'earth's axis' and the 'hour lines' are simply the intersections with the surface of the dial, of planes passing through the stile, which, with the 'plane of the meridian,' are inclined to one another at an angle of 15° in succession. The sun is always raised above its true place by refraction, the effect of this would be sensible when it is low down, but it is lost in the general indistinctness of the shadow when it is high.

Different kinds of Dials. There are several different kinds of 'dials.'

Horizontal. **Vertical.** When the 'plane of the dial' is 'horizontal,' it is called a *horizontal dial*, when it is 'vertical,' it is called a *vertical* or *erect dial*, and when the dial is both 'vertical' and 'perpen-

Prime Vertical. dicular to the meridian,' it is called a *prime vertical dial*, and when its 'plane' coincides with the 'meridian,' it may be called a

Meridian. '*meridian dial.*'

When the 'plane of the dial' is perpendicular to that of the meridian, but makes an oblique angle with the horizon

Inclining. it is called an *inclining dial.* When the dial is vertical, but inclined at an oblique angle to the meridian, it is called a

Declining. *declining dial.*

Dials may be constructed by means of a terrestrial globe, by dialling scales, or by stereographic projection, but the most accurate way embraces the principles of spherical trigonometry.

Construction of Horizontal Dial. To construct a horizontal dial.

Let S T_1 N T, fig. 72, be the plane of the dial, produced to cut the celestial sphere, P the pole, S P N the plane of the meridian and C P T the plane of an hour circle, that is, that at that particular time the sun was in the plane C P T_1 (higher or lower on the arc T_1 P according to the time of the year), and, therefore, casting C T as the

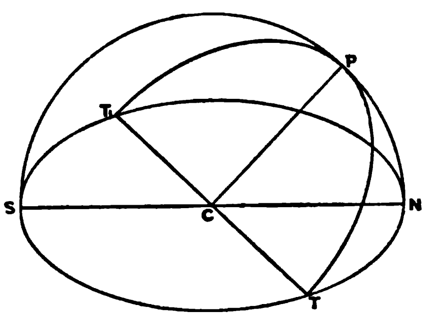


FIG 72.

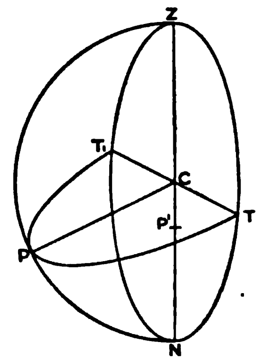


FIG. 73.

shadow of the stile C P, and marking off the arc N T round this circumference of the dial from the noon position of the shadow at C N. C P is the stile, and C N the direction of the sub-stile. P N is the latitude, T_1 C T is the hour line corresponding to the meridian, which gives the hours of the same name in the forenoon and afternoon, as for instance, five o'clock in the afternoon and morning, also S N is the hour line of twelve.

Let l = P N = latitude of place
h = angle T P N = hour angle in degrees.
t = N T = angular distance of the hour lines in succession from the hour line of noon.

Then in the triangle P N T, having the right angle at N :—

$$\sin l = \cot h \times \tan t$$
$$\therefore \quad \tan t = \sin l \tan h$$

or

$$\log \tan t = \log \sin l + \log \tan h - 10.$$

For any given latitude l, the above equation will give the values of t, when $h = 15°, 30°, 45°$, &c.

Example.—Supposing a horizontal dial is required for latitude 29° 52′.

Here, the angular distances of the hour lines from noon will be :—

For 1 p.m. or 11 a.m.—$\tan t = \sin 29° 52' \times \tan 15°$
$7° 36'$ $= \tan 7° 36'$
For 2 p.m. or 10 a.m.—$\tan t = \sin 29° 52' \times \tan 30°$
$16° 2'$ $= \tan 16° 2'$
For 3 p.m. or 9 a.m.—$\tan t = \tan 29° 52' \times \tan 45°$
$26° 28'$ $= \tan 26° 28'$

and so on.

On a brass disc of convenient diameter, mark off a diameter to represent the north and south line, and on each side of it mark off the hour lines at the estimated angles, *i.e.*, 7° 36′, 16° 2′, 26° 28′, &c. To this disc affix the gnomon, also of brass, the slant side of which must make an angle of 29° 52′ with the plane of the disc. Now, by means of a spirit level, carefully level in all directions some place, not liable to be shaken, so as to receive the metal disc. Then, having found the meridian, carefully make the north and south line of the disc coincide with it, and then finally secure the disc to its stand.

It should be noted that in the above, the gnomon is supposed to be a line, so that its mean shadow is cast by the centre of the sun, but if, as is usually the case, the gnomon be a plate, then the shadow of its edge is cast by the highest point of the sun, and therefore the shadow is about one minute ahead of its true position. Hence, the reading of the time will be one minute *too slow* before noon and the *same too* fast after noon.

Construction of Primo Vertical Dial.

To construct a prime vertical dial.

Let $T N T_1 Z$ (*see* fig. 73), be the plane of the dial, produced to cut the celestial sphere, P the pole of an opposite name to the latitude, Z P N the plane of the meridian, C P T the plane of an hour circle, and C P the direction of the stile. Then P N is the co-latitude, and $T C T_1$ is the hour line corresponding to that meridian, which gives the hours of the same name, at T in the forenoon, and at T_1 in the afternoon.

Let l = latitude of place.
$c = PN$ = co-latitude $= 90° - l$.
h = angle TPN = hour angle in degrees.
$t = NT$ = angular distance of the hour lines in succession from the hour line of noon.

Then, in the spherical triangle P N T, having the right angle at N :—

$$\cos l = \cot h \times \tan t$$
$$\therefore \quad \tan t = \cos l \tan h$$
$$\text{or} \quad \log \tan t = \log \cos l + \log \tan h - 10$$

Example.—To construct a vertical dial for a place in latitude 31° 30′.

Here, the angular distances of the hour lines from noon will be :—

For 1 p.m. or 11 a.m.—$\tan t = \cos 31^\circ 30' + \tan 15^\circ$
$12^\circ 52' \qquad = \tan 12^\circ 52'$ (nearly)
For 2 p.m. or 10 a.m.—$\tan t = \cos 31^\circ 30' + \tan 30^\circ$
$26^\circ 12' \qquad = \tan 26^\circ 12'$
For 3 p.m. or 9 a.m.—$\tan t = \cos 31^\circ 30' + \tan 45^\circ$
$40^\circ 27' \qquad = \tan 40^\circ 27'$

and so on.

The angle formed by the gnomon with the vertical face of the dial will, of course, be 58° 30′ = co-latitude.

In a horizontal and vertical dial, the *elevation* of the style is respectively equal to the *latitude* and *co-latitude* of the place, for in fig. 72, showing a horizontal dial, P C N is the latitude, and in fig. 73, showing a vertical dial, P C N is the co-latitude.

The inclination of the plane of an oblique dial to the horizon (or plane of a horizontal dial), is called its *inclination,* and its inclination to the prime vertical its *declination.*

In this manner a dial constructed for one place may be used at another. For instance, suppose a 'horizontal dial' made for Delhi (lat. 28° 39′) is to be set up in Lahore (lat. 31° 34′) so as to show Lahore time. The difference in latitude is 2° 55′. The stand to receive the dial will not now be horizontal, but will be inclined towards the north at an angle of 2° 55′ with the horizon. It will then show Lahore time.

Dials at the Equator.

At the Equator, the stile and sub-stile of a horizontal dial coincide. The stile then, has to be placed above the plane of the dial, and parallel to the meridian. The hour lines are now all parallel with the north and south line, and distant from it by the tangent of the corresponding hour circle's inclination to the meridian, the height of the stile being the radius.

Let S = height of stile above dial.
h = hour angle in degrees.
t = perpendicular distance of the corresponding hour line from that of noon.

Then

$$t = S \tan h.$$

The stile of a 'prime vertical dial' would evidently be perpendicular to its plane, and the hour lines would make angles of 15° with each other in succession.

Oliver's Mean Time Dial.

In order to read *mean* time, Major-General Oliver has invented a 'sun-dial' (*see* fig. 74), by which the time is indicated, not by the shadow of a straight edge, but by the point where an equatorial circular line is cut by the edge of the shadow of a curved surface, the curvature of which is so arranged as to compensate for the 'equation of time.'

The instrument is a universal one, and consists of a meridional semicircle, the diameter of which is an axis carrying the curved stile, and an equatorial circular arc. The latter has engraved upon its concave surface a graduated line on which are marked the hours and their subdivisions. There is a screw for clamping the meridional arc at the proper position for any given latitude, and another clamp for adjusting the equatorial arc.

The dial not only indicates local mean time, but by a very simple adjustment may be set so as to show any required standard time. Thus it might be set at Plymouth to indicate Greenwich time.

Strictly speaking, there ought to be two stiles, one to be used from June to December, and the other from December to June. But by adopting a stile of

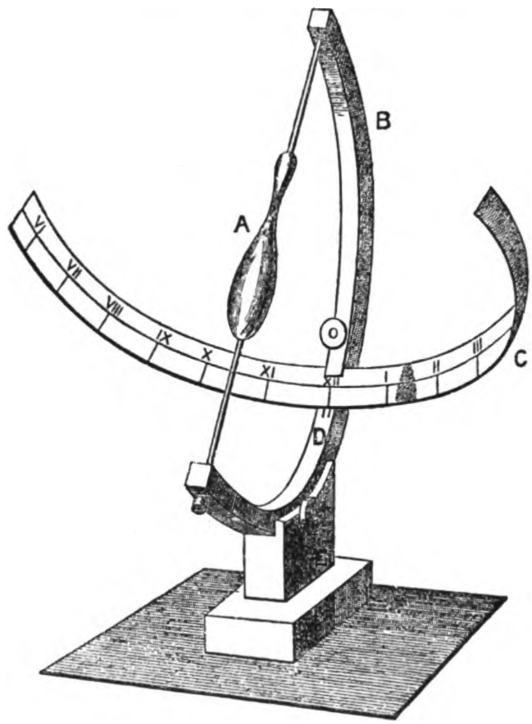


FIG. 74.

mean contour the greatest error introduced at any time is only about one minute, an amount not exceeding the error due to the softness of the edge of the shadow.

Directions for Use.

The dial consists of a latitude arc B, carrying a stile A, an hour circle C, and a stand E.

The instrument is first put together, and the arc B clamped by its screw at the correct position for the latitude of the place. The thin end of the stile must point to the north, and in the northern hemisphere will be uppermost. South of the equator the thick end will be uppermost.

The stand must be level, and its sides must lie north and south, and east and west.

The local time is indicated by the point where one of the edges of the shadow of the stile A, cuts the central line of the hour circle. The latter is divided into hours, and each hour into twelve spaces of five minutes.

Four times a year the equation of time vanishes, and the stile would then intersect its own axis. To allow for the necessary thickness of the latter a slight adjustment of the hour circle is necessary at these times.

At the point D, on the arc B, are two marks. The hour circle must be clamped with the 12 o'clock line coinciding with one of these marks. The left-hand or west mark is used from about the 15th April to the 21st June, and from about the 25th August to the 21st December. At other times the 12 o'clock line must coincide with the right-hand or east mark. If it coincides with the left one (looking north) the left-hand edge of the shadow is the one which indicates the time. When the right mark is used, the right side of the shadow is read. Thus, the 12 o'clock line must be shifted from one mark to the other four times a year, viz. about the 15th April, 21st June, 25th August and 21st December. The exact date is of no consequence.

If the dial is wanted to indicate a certain standard time, instead of local mean time, the hour circle is unclamped and shifted through the arc corresponding to the difference of longitude. Thus, if we wanted to set it at Dublin (which is 24 minutes in time west of London) to indicate London time, we should set the 24 minutes past 12 mark on the hour circle to coincide with the mark at D.

A simple approximate method of determining the meridian is described on page 29, Part I.

CHAPTER XV.

GAUGING STREAMS AND RIVERS.

Object of Chapter.

THOUGH the measurement of running water is scarcely a branch of surveying, yet, as the surveyor may in connection with his more legitimate work, find it convenient to be in a position to determine the volume of water discharged by streams and rivers (in the case of surveys for water-supply, irrigation, or river improvements), it seems desirable to introduce a few notes on this subject.

Gauging small Streams.

The most usual method of gauging a small stream is to obstruct its course by means of a weir, constructed of planks, or (if the observations are to be continued over a prolonged period) of masonry, or concrete. In this way a still pond is formed in the bed of the stream, from which the water is discharged, in a cascade of regular form and known dimensions, through a V-shaped or rectangular notch in a thin plate fixed to the crest of the weir. The rate of discharge is then ascertainable, by two measurements only, viz. the depth of the water in a state of rest, above the bottom of the notch or sill of the depression, and the length of the opening, in either case. These dimensions being known, the discharge is calculated by well-known formulæ, and by employing coefficients, appropriate to the form and conditions of the notch or sill. To ensure accuracy it is essential that the form and general conditions of the notch or sill, over which the water flows, should coincide with one or other of the forms that have been the subject of experiment, and whose coefficients are therefore thoroughly ascertained.

The most usual form of gauge (except for very small flows, when the V-shaped notch is convenient) is the rectangular notch, cut in a thin plate, preferably of metal, having a horizontal sill and vertical sides, the plane of the plate being vertical, and smooth on the up stream side. The thickness of the edge, over which the water flows, should not exceed about $\frac{1}{8}$ inch, and should be cut true and square to the surface of the plate, and with a sharp arris. If these conditions be attended to, it will be found that the under-surface of the sheet of water or 'apron,' passing over the edge, rises from the sill, not even wetting its upper surface. A well-conditioned cascade is thus formed, one whose coefficient has been accurately ascertained.

M. Bazin, in his exhaustive treatise 'Sur l'Écoulement d'Eau en Déversoir' (from which much of what follows is taken), gives the following measurements to the *upper and lower water surfaces* of the cascade, *below* the 'still water level' and *above* the 'sill level' respectively (*vide* fig. 75).

In the first place, careful observation will show, that at a certain distance up-

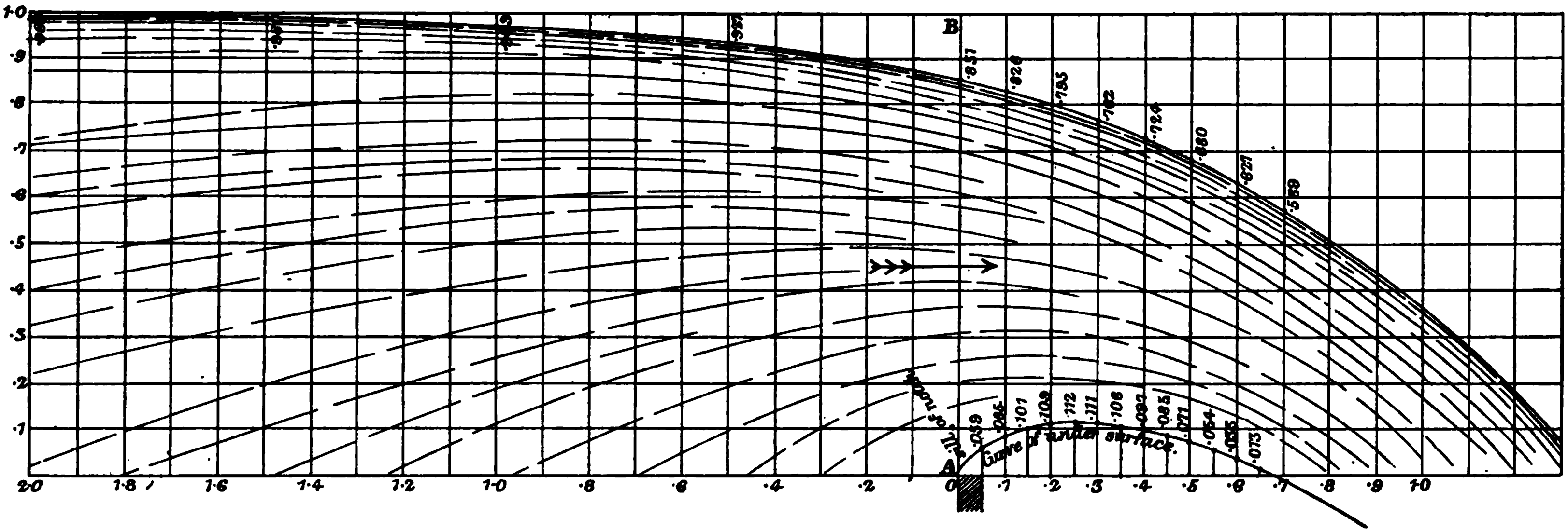
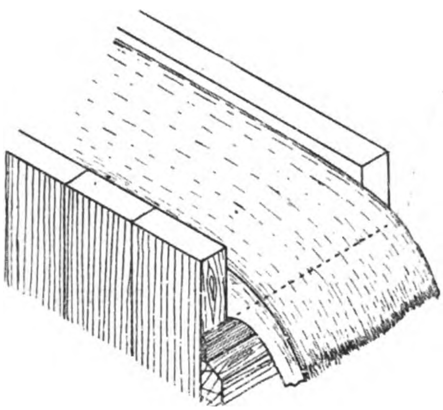

A
B
Curve of under surface.
·851
·826
·785
·762
·724
·680
·621
·589
·059
·085
·101
·109
·112
·111
·108
·099
·085
·071
·054
·035
·013

FIG 75.

stream from the sill, the water surface of the approaching stream begins to fall below the general surface of still water, the inclination of the surface becoming greater as the water approaches the crest of the fall. On the crest itself, the depth of the water over the sill is but 0·85 of the height of the still water surface. After passing the sill, the water surface falls rapidly, in an approximately parabolic curve. The under surface of the cascade forms an arch the maximum height of which is about one-tenth of the head (h) measured from '*sill*' to '*still water*' level, and is at a distance of about $0 \cdot 3 \times h$ from the sill, measured from its up-stream arris. This surface regains the level of the sill, at about $0 \cdot 7 \times h$ from its up-stream arris.



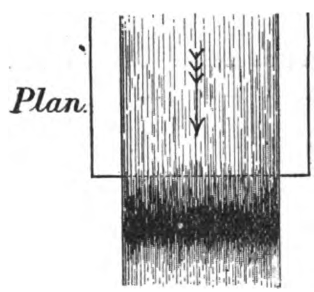


FIG. 76.

According to M. Bazin, the figure of the stream section is nearly constant with all heads, the co-ordinates being a constant multiple of h.

Notch without 'End Contractions.'

The form of vertical section, shown in fig. 76, affords the best results, but is seldom obtainable in practice, unless the gauging station is intended to be of a more or less permanent character.

The desired section can only be established throughout the whole length of a notch, when the approaching water is guided by vertical planes, coinciding with the extremities of the notch, that is to say, when the sill extends from side to side of a rectangular channel (*vide* fig. 76).

A notch arranged in this manner is said to be '*without end contractions.*'

To establish the desired overfall or vertical section, it is essential that air should have free access to the under side of the apron of water.

Forms of Overflow.

If the width of the channel below the notch be the same as that of the notch, the sides of the apron will adhere to the sides of the channel and a partial vacuum will be formed beneath the apron, whilst the full atmospheric pressure acts upon its upper surface (*vide* fig. 77). The excess of pressure on the upper surface produces a depression of the stream-lines, and a greater discharge, for a given head is afforded, than would result if air were admitted freely to the underside of the apron.

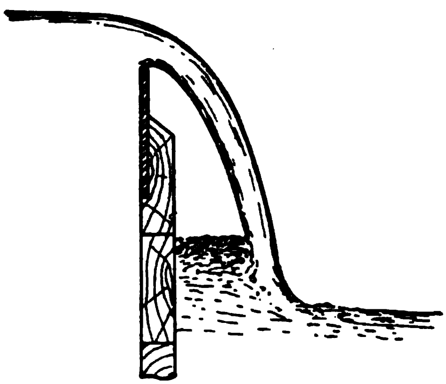


FIG. 77.

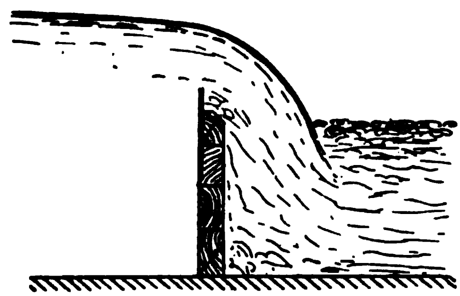


FIG. 78.

When the drop below the apron is small in comparison with the head, the space between the notch board and the apron may fill up solid (*vide* fig. 78). In such a case, the discharge may be as much as 28 per cent. in excess of that given by the formula for a free apron.

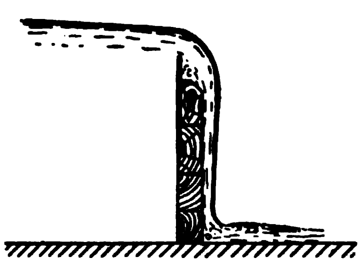


FIG. 79.

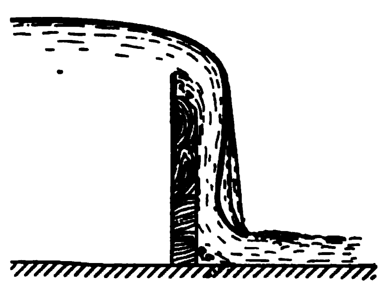


FIG. 80.

It is, therefore, important that the drop below the sill should be greater than the head, for if less, the apron is apt to adhere to the notch board, more especially if the notch be long relatively to the head.

Between the limits of an apron, to the underside of which air has free access, and one which adheres completely to the notch board, several forms may occur.

(1) Depressed apron (*vide* fig. 77).

(2) Adhering apron for a small head (*vide* fig. 79).
(3) Adhering apron for a large head (*vide* fig. 80).
(4) Apron adhering at crest but detached beneath (*vide* fig. 81).
(5) Apron adhering below, detached at crest (*vide* fig. 82).

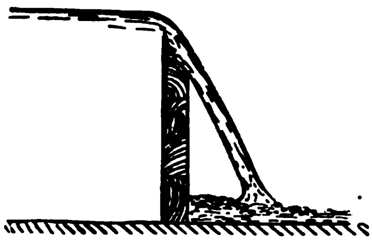


FIG. 81.

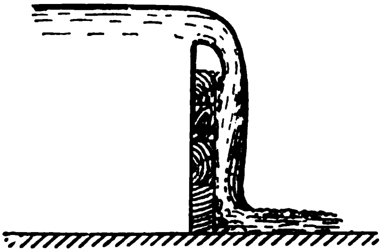


FIG. 82.

(6) Apron solid underneath (*vide* fig. 83).
(7) Adhesion of apron with tail-water headed up (*vide* fig. 84).
(8) Solid apron with high tail-water (*vide* fig. 78).

Reliable results cannot be obtained under any of the circumstances described above from (1) to (8), nor will the coefficient be constant for any one form under varying conditions of head.

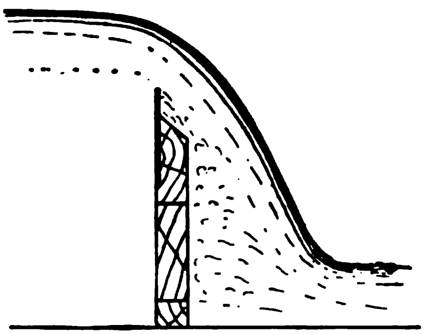


FIG. 83.

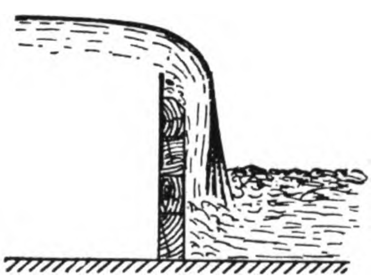


FIG. 84.

If the notch board be other than vertical, the coefficients applicable to a vertical notch board do not apply, nor are new coefficients readily calculable to meet the altered conditions.

Notch with 'Complete End Contraction.'

It is frequently necessary to gauge with a notch in such a position that the approaching stream or flow from the artificially formed pool is wider than the notch.

Under these circumstances the flow is contracted, both vertically, between the sill and the crest of the apron, and horizontally, between the two vertical ends of the notch. The stream lines curve

inwards at the ends, and the length of the apron is less than that of the notch (*vide* fig. 85).

The inward stream curves materially diminish the discharge of a notch with 'complete end contractions,' as compared with one of the same length without 'end contractions.'

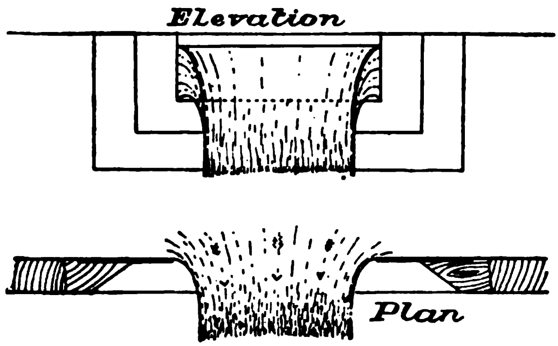


FIG. 85.

The relation of 'end contractions' to the head, does not appear to have been fully investigated scientifically, but an empirical formula which, within limits, agrees with experiment, will be given later.

To avoid the uncertainty due to the effect of 'end contractions,' a V-notch is often used. The most usual form is a rectangular V, fig. 86, having a breadth equal to twice the vertical height.

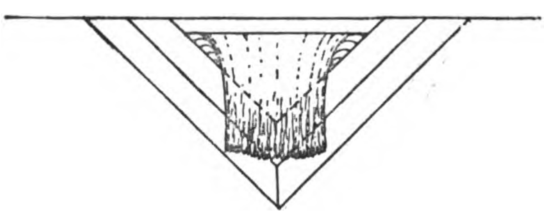


FIG. 86.

Since the figure of the overfalling stream is practically the same for all heights, it is probable that the effect of contraction is a constant proportion for all depths. For these reasons a V notch is to be preferred to one of rectangular form, where it can be safely used, but it must be remembered that the distance between the bottom of the V and the water below the gauge board, must in this case be greater for a given volume of water than with the rectangular notch.

A single table suffices to give the discharge of any 'rectangular V notch' through a wide range of heads, and the same table will apply to all V notches if a suitable *multiplier* be employed.

The following notation will be used throughout.

Let h = head of water in feet, measured to surface of still water at some distance above, or to one side of the notch (a common practice is to make this distance 10 feet, which may be unnecessarily great, though if the head be considerable, too little).

l = length of the notch in feet.

d = depth from sill to bottom of channel, if rectangular, otherwise to the bed of the natural stream.

f = fall from sill to surface of water beneath the overfall.

Q = discharge per second in the unit measuring h and l (say feet).

g = acceleration due to gravity in feet per second.

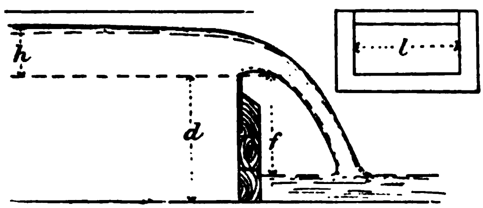


FIG. 87.

Then, the primitive formula, used in England is

$Q = \frac{2}{3} c l h \sqrt{2 g h}$, when c is the coefficient of contraction, as determined by experiment, approximately $0 \cdot 622$.

French writers adopt a coefficient μ instead of $\frac{2}{3} c$. The expression given becomes $Q = \mu l h \sqrt{2 g h}$. This involves no assumption as to the relative velocities of stream at different depths. Bazin gives us the result of his experiments $\mu = 0 \cdot 405 + \frac{0 \cdot 0098}{h}$ which diminishes slightly as the head increases. Assuming foot-second units, we may put $\sqrt{2 g} = 8 \cdot 02$. The first formula then becomes $Q = 3 \cdot 33\, l h^{\frac{3}{2}}$ cubic feet per second (Francis' formula), and the second $Q = \left(3 \cdot 248 + \frac{\cdot 0786}{h}\right) l h^{\frac{3}{2}}$ cubic feet per second (Bazin's formula).

TABLE A.

ACCORDING TO M. BAZIN.

Head in inches	1	2	3	4	5	6	7
Value of $\mu \times \sqrt{2g}$	3·405	3·383	3·366	3·353	3·342	3·334	3·327
Head in inches	8	9	10	11	12	13	14
Value of $\mu \times \sqrt{2g}$	3·300	3·298	3·295	3·293	3·291	3·289	3·287

The value of $\mu \times \sqrt{2g}$ up to 14″ head, as calculated by the formula above, is shown in the preceding Table A.

Allowance for 'End Contractions' in Rectangular Notches.

Francis gives the following expression for the effect of 'end contractions' in rectangular notches.

$$Q = 3{\cdot}33\ (l - 0{\cdot}1\ n\ h)\ h^{\frac{3}{2}}$$

when n = number of 'end contractions.'

In the common case of two 'end contractions'

$$Q = \left(l - \frac{h}{5}\right) \times 3{\cdot}33\ h^{\frac{3}{2}}.$$

This was found to give good results, corresponding well with experiment, so long as l was not less than $3h$.

The expression appears to be of limited application, for if $h = 5\,l$ the expression between brackets becomes zero, and the discharge also zero, which is absurd.

The attached Table B, gives the discharge in imperial gallons per day for a notch 1 inch long, *without* 'end contractions,' using the formula of Francis. Table C, gives the *deductions* to be made with respect to two 'end contractions' for various values of h, in inches.

For heads greater than 11·95 inches but less than 47·8 inches, *divide the given head by* 4, *and enter the tables with the head nearest to the result*, *multiplying the number* thus obtained from Table B by 8, and that from Table C by 32. The results will be the required discharge in gallons per day.

Example.

Let h = 19·75 inches and l = 10 feet,

then 19·75 ÷ 4 = 4·94 nearly, say 4·95.

From Table B, 39560 is obtained.

$$39560 \times 8 \times 10 \times 12 = 37970000 \text{ nearly.}$$

From Table C, $39160 \times 32 = 1250000$

∴ Total discharge $= 36720000$ gallons per day.

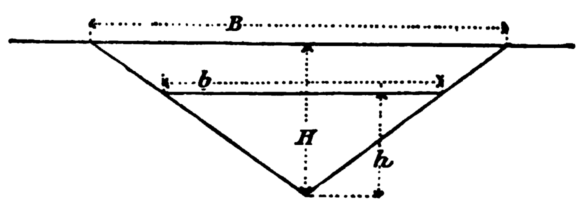


FIG. 88.

Formula for V Notch.

The general formula for the discharge from a V notch is as follows.

Let

H = the vertical height of notch from bottom to top of V (*vide* fig. 88)

and B = its greatest breadth.

b = breadth at level of still water in feet.

TABLE B.—RECTANGULAR NOTCH DISCHARGE IN IMPERIAL GALLONS PER DAY, PER INCH IN LENGTH, WITHOUT END CONTRACTIONS.

Head in inches.	0	1	2	3	4	5	6	7	8	9	10	11	Head in inches.
0	0	3592	10160	18660	28740	40160	52790	66530	81280	96990	113600	131000	0
·05	40·16	3865	10543	19130	29280	40760	53450	67240	82040	97800	114400	131900	·03
·1	113·6	4144	10931	19610	29820	41370	54120	67960	82810	98610	115300	132800	·1
·15	208·7	4430	11324	20080	30370	41980	54790	68680	83580	99420	116200	133700	·15
·2	321·3	4722	11721	20560	30920	42590	55460	69400	84350	100250	117000	134600	·2
·25	449·0	5020	12123	21050	31470	43210	56130	70120	85120	101100	117900	135500	·25
·3	590·2	5324	12530	21530	32030	43830	56800	70850	85890	101900	118700	136400	·3
·35	743·8	5634	12940	22020	32590	44450	57480	71580	86670	102700	119600	137300	·35
·4	908·7	5950	13360	22520	33150	45070	58160	72310	87450	103500	120500	138300	·4
·45	1084	6272	13780	23020	33720	45700	58840	73040	88230	104300	121300	139200	·45
·5	1270	6599	14200	23520	34290	46330	59530	73780	89020	105200	122200	140100	·5
·55	1465	6932	14630	24020	34860	46970	60220	74520	89800	106000	123100	141000	·55
·6	1669	7270	15060	24540	35440	47600	60910	75260	90590	106800	124000	141900	·6
·65	1882	7613	15500	25050	36020	48240	61600	76000	91380	107700	124800	142800	·65
·7	2104	7962	15940	25570	36600	48880	62290	76750	92180	108500	125700	143800	·7
·75	2333	8316	16380	26090	37190	49530	62990	77500	92970	109400	126600	144700	·75
·8	2570	8675	16830	26610	37780	50180	63700	78250	93770	110200	127500	145600	·8
·85	2815	9039	17280	27130	38370	50830	64400	79000	94570	111000	128400	146500	·85
·9	3067	9407	17740	27660	38960	51480	65110	79760	95370	111900	129300	147500	·9
·95	3326	9781	18200	28200	39560	52130	65820	80520	96180	112700	130200	148400	·95
	0	1	2	3	4	5	6	7	8	9	10	11	

TABLE C.*—ALLOWANCE (IN GALLONS PER DAY) FOR TWO END CONTRACTIONS (TO BE DEDUCTED FROM TABLE B).

NOTE.—These corrections only apply when h is less than $\frac{l}{3}$

Head in inches.	0	1	2	3	4	5	6	7	8	9	10	11	Head in inches.
0	..	718·4	4064	11200	22990	40160	63350	93140	130000	174600	227200	288300	0
·05	·4016	811·6	4323	11670	23710	41170	64680	94810	132100	177000	230000	291600	·05
·1	2·272	911·7	4591	12160	24450	42200	66020	96500	134100	179500	232900	294900	·1
·15	6·260	1019	4869	12650	25210	43240	67380	98210	136200	181900	235800	298200	·15
·2	12·85	1133	5157	13160	25970	44300	68760	99930	138300	184400	238700	301600	·2
·25	22·45	1255	5455	13680	26750	45370	70160	101700	140400	187000	241600	305000	·25
·3	35·41	1384	5764	14210	27550	46460	71570	103400	142600	189500	244600	308400	·3
·35	52·06	1521	6082	14760	28350	47560	73000	105200	144700	192000	247600	311800	·35
·4	72·70	1666	6411	15310	29170	48680	74440	107000	146900	194600	250600	315200	·4
·45	97·59	1819	6750	15880	30010	49820	75910	108800	149100	197200	253600	318700	·45
·5	127·0	1980	7099	16460	30860	50970	77390	110700	151400	199800	256700	322200	·5
·55	161·2	2149	7460	17060	31780	52130	78880	112500	153600	202500	259700	325700	·55
·6	200·3	2326	7831	17670	32600	53310	80400	114400	155800	205100	262800	329200	·6
·65	244·7	2512	8213	18290	33490	54510	81930	116300	158100	207800	265900	332800	·65
·7	294·5	2707	8606	18920	34410	55730	83480	118200	160400	210500	269000	336400	·7
·75	350·0	2911	9010	19560	35330	56960	85040	120100	162700	213200	272200	340000	·75
·8	411·2	3123	9425	20220	36260	58200	86620	122100	165000	216000	275400	343600	·8
·85	478·5	3344	9851	20890	37220	59470	88239	124000	167400	218700	278600	347300	·85
·9	552·1	3575	10290	21580	38180	60740	89850	126000	169800	221500	281800	350900	·9
·95	631·9	4815	10740	22280	39160	62040	91480	128000	172200	224400	285000	354600	·95
	0	1	2	3	4	5	6	7	8	9	10	11	

* This table is at first sight anomalous, for, referring to the table on page 227, it will be seen that for a notch 1" long under a 12" head, the discharge, without end contractions would be only 148,400 gals. per day. Whereas, by this table, the deduction for end contractions would be 354,600 gals. per day, or more than the whole discharge. The formulæ do not apply when the length is less than three times the head. Thus for a 12" head, the shortest notch to which this table could be applied would be 36", for a 6" head 18", and so on.

Then $Q = C \times \frac{1}{2} b h \times k \sqrt{2gh}$ in cubic feet per second.

Where C = coefficient of contraction, which may be taken at $0{\cdot}617$.

and k = the ratio of maximum velocity at apex, to mean velocity $= \frac{8}{15}$.

$\sqrt{2g} = 8{\cdot}02$.

Hence $Q = 0{\cdot}617 \times \frac{bh}{2} \times \frac{8}{15} \times 8{\cdot}02 \sqrt{h}$

or $Q = 1{\cdot}32 b h^{\frac{3}{2}}$.

For a rectangular V notch $b = 2h$, and $Q = 2{\cdot}639 h^{\frac{5}{2}}$ cubic feet per second, h being in feet.

The following résumé gives the formulæ reduced to different denominations, and Table D gives the discharge from rectangular V notches in gallons per day.

(A) Francis' formula for discharge Q (rectangular notch)

(1) in c. ft. per minute,	$Q = 200\, l h^{\frac{3}{2}}$	l = length. h = head in feet.
(2) in gals. ,,	$Q = 1250\, l h^{\frac{3}{2}}$	
(3) in gals. per day	$Q = 1795000\, l h^{\frac{3}{2}}$ or $1800000\, l h^{\frac{3}{2}}$ nearly	

(B) Francis' correction for two end contractions (rectangular notch), to be deducted.

(1) in c. ft. per minute,	$40\, h^{\frac{5}{2}}$	h = head in feet.*
(2) in gals. ,,	$250\, h^{\frac{5}{2}}$	
(3) in gals. per day	$359000\, h^{\frac{5}{2}}$ or $360000\, h^{\frac{5}{2}}$ nearly	

(C) Bazin's formula for discharge, page 225 (rectangular notch) †

(1) in c. ft. per minute,	$194{\cdot}9\, l h^{\frac{3}{2}} + 4{\cdot}72\, l h^{\frac{1}{2}}$	l length. h = head in feet.
(2) in gals ,,	$1215\, l h^{\frac{3}{2}} + 29{\cdot}4\, l h^{\frac{1}{2}}$	
(3) in gals. per day	$1749000\, l h^{\frac{3}{2}} + 42300\, l h^{\frac{1}{2}}$	

(D) Formula for rectangular V notch

(1) in c. ft. per minute,	$Q = 158{\cdot}3\, h^{\frac{5}{2}}$	h = head in feet. For other V notches multiply by $\frac{\text{base}}{2 \times \text{height}}$.
(2) in gals. ,,	$Q = 986\, h^{\frac{5}{2}}$	
(3) in gals. per day	$Q = 1421000\, h^{\frac{5}{2}}$	

The height above the crest of the weir h must be measured to still water, and the formula supposes that the pond above the weir is still, and that there is no velocity of approach (or a negligible quantity), and this may generally be secured. It is only necessary to make the pond sufficiently wide and deep, since if its width be double the length of the notch, still water will be found at the side near the ends of the board.

Should the velocity of approach be so much as 1 foot per second, there will be a reduction in the value of h to the extent of $0{\cdot}192$ of an inch and the calculated discharge will be too small by the amount due to this head.

* The writer has found by tank measurement that Francis' formula gave excellent results with a notch one foot long and head = 2″ to 3″.

† M. Bazin's experiments were made with heads exceeding 4″, and therefore for lower heads his coefficient must be accepted with caution.

TABLE D.—DISCHARGES OF RECTANGULAR V NOTCHES IN GALLONS PER DAY

Head in inches.	0	1	2	3	4	5	6	7	8	9	10	11	Head in inches.
0	..	2847	16100	44380	91090	159100	251000	369000	515300	691700	900200	1142000	0
·05	1·591	3216	17130	46250	93970	163100	256300	375700	523400	701400	911500	1155000	·05
·1	9·002	3613	18190	48170	96890	167200	261600	382400	531600	711100	922900	1169000	·1
·15	24·81	4037	19290	50130	99880	171300	267000	389100	539800	720900	934300	1182000	·15
·2	50·92	4490	20440	52150	102900	175500	272500	396000	548100	730800	945900	1195000	·2
·25	88·96	4973	21620	54210	106000	179800	278000	402900	556500	740800	957500	1208000	·25
·3	140·3	5485	22840	56310	109100	184100	283600	409900	565000	750800	969200	1222000	·3
·35	206·3	6028	24100	58470	112300	188500	289200	416900	573500	761000	981000	1235000	·35
·4	288·1	6602	25400	60680	115600	192900	295000	424100	582200	771200	992900	1249000	·4
·45	386·7	7707	26750	62930	118900	197400	300800	431300	590900	781500	1005000	1263000	·45
·5	503·2	7845	28130	65240	122300	202000	306600	438500	599600	791900	1017000	1277000	·5
·55	638·6	8515	29560	67600	125700	206600	312600	445900	608500	802300	1029000	1291000	·55
·6	793·8	9218	31030	70000	129200	211300	318600	453300	617400	812900	1041000	1305000	·6
·65	969·7	9950	32540	72460	132700	216000	324600	460800	626400	823500	1054000	1319000	·65
·7	1167	10730	34100	74960	136300	220800	330800	468300	635500	834200	1066000	1333000	·7
·75	1387	11530	35710	77520	140000	225700	337000	476000	644700	845000	1079000	1347000	·75
·8	1630	12370	37350	80130	143700	230600	343300	483700	654000	855900	1091000	1362000	·8
·85	1896	13250	39030	82790	147500	235600	349600	491500	663300	866800	1104000	1376000	·85
·9	2187	14170	40770	85510	151300	240700	356000	499400	672700	877900	1117000	1391000	·9
·95	2504	15120	42550	88270	155200	245800	362500	507300	682200	889000	1129000	1405000	·95
	0	1	2	3	4	5	6	7	8	9	10	11	

If the velocity be ½ foot per second, the reduction 0·192 of an inch must be divided by 4, and if 2 feet per second, it must be multiplied by 4.*

Drowned Weirs. M. Bazin has extended his experiments, so as to give coefficients for what is known as a drowned weir or notch, i.e. one in which the tail water below the fall, is above the crest of the weir.

In this case fig. 89, the head must be measured below as well as above the notch, the second or down-stream measurement being taken at a point at which the tail water has resumed steady motion, that is, below the standing waves which occur on the down-stream side of the notch, and pre-supposing that the channel below the weir is level and regular in section.

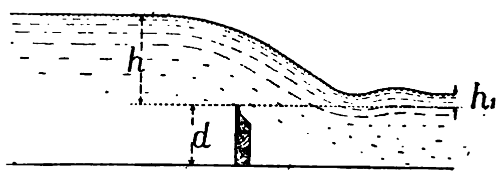


FIG. 89.

Table E, gives (according to M. Bazin) the ratios of the discharges from solid aprons and drowned weirs, to those which would be obtained with the same head, but with the typical form of apron as shown in fig. 75.

In the table

	m	represents	the	actual coefficient of discharge
	m_1	,,	,,	corresponding coefficient with the typical apron
	h	,,	,,	actual head to still water up-stream
	h_1	,,	,,	actual head to still water down-stream, both measured above the crest
and	d	,,	,,	height of the dam itself.

Thus, for the ordinary solid apron (fig. 79), the tail water has no effect, and the ratio $\frac{h_1}{d}$ does not enter into the result.

For the solid apron with heaped up tail water, the level down-stream is below the crest. Hence h_1 and $\frac{h_1}{d}$ are negative (fig. 78).

To use the table, the coefficient of discharge for a given head, should be taken from the table on page 225, multiplied by the proper factor from the annexed table, and substituted in the formula on page 225.

Or the discharge may be computed as for a typical apron by any of the formulæ on page 225 (or taken from the tables A and B, pp. 227, 230) and then multiplied by the proper factor from the table.

It is important that the measurement of h should be made with as great accuracy as is practicable. An ordinary graduated scale immersed in water is not satisfactory. The water level in the vicinity of the scale is raised by capillary attraction, so that it is not easy to estimate the true level.

* *Vide* Appendix J, for further remarks on velocity of approach.

TABLE E.

Values of $\frac{h}{d}$.	Values of $\frac{m}{m_1}$ for																								
	Solid apron (fig. 79)	Solid apron with heaped up tail water $\frac{h_1}{d} = -$								Drowned notch with $\frac{h_1}{d} = +$															
		0·35	0·30	0·25	0·20	0·15	0·10	0·05	0	0·05	0·10	0·20	0·30	0·40	0·50	0·60	0·70	0·80	0·90	1·00	1·10	1·20	1·30	1·40	1·50
0·10	..	..	..	..	..	..	..	..	..	0·81	0	..	..	..	..	..	..	..	..	..	..	..	..	..	..
0·15	..	..	..	..	..	..	..	1·06	1·01	0·90	0·76	..	..	..	..	..	..	..	..	..	..	..	..	..	..
0·20	..	..	..	..	..	..	1·10	1·06	1·02	0·94	0·86	0	..	..	..	..	..	..	..	..	..	..	..	..	..
0·25	..	..	..	..	1·15	1·12	1·09	1·06	1·03	0·97	0·91	0·65	..	..	..	..	..	..	..	..	..	..	..	..	..
0·30	..	..	..	1·16	1·14	1·11	1·09	1·06	1·03	0·99	0·94	0·77	0	..	..	..	..	..	..	..	..	..	..	..	..
0·35	..	1·18	1·16	1·14	1·13	1·10	1·08	1·06	1·04	1·00	0·96	0·84	0·59	..	..	..	..	..	..	..	..	..	..	..	..
0·40	1·19	1·17	1·15	1·14	1·12	1·10	1·08	1·06	1·04	1·01	0·98	0·89	0·71	0	..	..	..	..	..	..	..	..	..	..	..
0·45	1·16	..	1·14	1·13	1·11	1·10	1·08	1·06	1·04	1·02	0·99	0·91	0·79	0·55	..	..	..	..	..	..	..	..	..	..	..
0·50	1·13	..	..	1·12	1·11	1·09	1·08	1·06	1·04	1·02	1·00	0·93	0·84	0·67	0	..	..	..	..	..	..	..	..	..	..
0·55	1·11	..	..	..	..	1·09	1·08	1·06	1·05	1·03	1·01	0·95	0·87	0·75	0·53	..	..	..	..	..	..	..	..	..	..
0·60	1·09	..	..	..	..	..	1·07	1·06	1·05	1·03	1·01	0·96	0·90	0.80	0·64	0	..	..	..	..	..	..	..	..	..
0·65	1·08	..	..	..	..	..	..	1·06	1·05	1·03	1·02	0·98	0·92	0·84	0·72	0·51	..	..	..	..	..	..	..	..	..
0·70	1·06	..	..	..	..	..	..	..	1·05	1·04	1·02	0·98	0·94	0·87	0·77	0·62	0	..	..	..	..	..	..	..	..
0·75	1·05	..	..	..	..	..	..	..	..	1·04	1·03	0·99	0·95	0·89	0·81	0·70	0·49	..	..	..	..	..	..	..	..
0·80	1·04	..	..	..	..	..	..	..	..	..	1·03	1·00	0·96	0·92	0·84	0·75	0·60	0	..	..	..	..	..	..	..
0·85	1·03	..	..	..	..	..	..	..	..	..	..	1·01	0·97	0·93	0·87	0·79	0·68	0·47	..	..	..	..	..	..	..
0·90	1·02	..	..	..	..	..	..	..	..	..	..	1·01	0·98	0·94	0·89	0·82	0·73	0·59	0	..	..	..	..	..	..
0·95	1·01	..	..	..	..	..	..	..	..	..	..	..	0·99	0·96	0.91	0·85	0·77	0·66	0·46	..	..	..	..	..	..
1·00	1·01	..	..	..	..	..	..	..	..	..	..	..	1·00	0·97	0·93	0·88	0·81	0·72	0·58	0	..	..	..	..	..
1·05	1·00	..	..	..	..	..	..	..	..	..	..	..	..	0·98	0·94	0·90	0·84	0·76	0·65	0·46	..	..	..	..	..
1·10	0·99	..	..	..	..	..	..	..	..	..	..	..	..	0·98	0·95	0·91	0·86	0·79	0·70	0·57	0	..	..	..	..
1·15	0·99	..	..	..	..	..	..	..	..	..	..	..	..	..	0·96	0·93	0·88	0·82	0·75	0·64	0·45	..	..	..	..
1·20	0·98	..	..	..	..	..	..	..	..	..	..	..	..	..	0·97	0·94	0·90	0·85	0·78	0·69	0·56	0	..	..	..
1·25	0·98	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0·95	0·91	0·87	0·81	0·74	0·63	0·44	..	..	..
1·30	0·97	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0·96	0·92	0·89	0·84	0·77	0·69	0·55	0	..	..
1·35	0·97	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0·94	0·90	0·86	0·80	0·73	0·62	0·44	..	..
1·40	0·96	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0·95	0·91	0·88	0·83	0·77	0·68	0·55	0	..
1·45	0·96	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0·92	0·89	0·85	0·80	0·72	0·62	0·43	..
1·50	0·96	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0·94	0·90	0·86	0·82	0·76	0·67	0·54	0

If a graduated scale be used, it should be rubbed with French chalk before being immersed in water.

The most accurate method of gauging the surface level, is by means of a fish or other hook attached to a rod or wire, and weighted to keep the wire strained. If the hook be immersed in the water and then slowly raised, the point will show at the exact moment of its leaving the water.

It may frequently be desirable to maintain a notch gauge for some considerable time, and to take many readings of the level of the water. For this purpose, a float may be used to which a vertical scale is attached, the height of the float above datum being read by inspection (*vide* fig. 90).

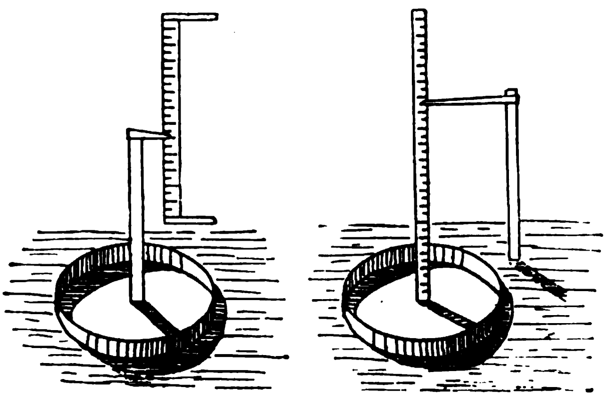


FIG. 90.

If continuous readings be desired, the float may be attached to a movable index, carrying a pin bearing on a ruled scroll of paper rolled round a drum, and actuated by clockwork, so as to make a complete revolution in 24 hours or in 7 days.

The accuracy of such a record must depend on the pin point touching zero on the paper scroll, when the water is still and level with the top of the weir, but is not running over.

The float should be of large diameter as compared with its depth in order to secure great steadiness and rapidity of action, and it should, if possible, be enclosed in a pipe of stoneware or cast iron, the lower orifice of which is bent to a horizontal direction and is in still water unaffected by the velocity of the stream. This may be best secured by placing it in a sump below the bed of the stream, and towards one side of it.

Stepped Notch.

Should it be necessary in using the notch gauge to measure floods as well as dry-weather flow (which may at its least be $\frac{1}{500}$ or even $\frac{1}{1000}$ of the highest flood) it is necessary to use a gauge made in steps, fig. 91, in order that the dry-weather flow (which is the really important item) may be measured with accuracy, and that it shall also be possible to estimate the volume of the flood water.

It will be easily understood that if the depth of the overfall, from the edge of

the notch to the level of water below the notch, be sufficient when the stream is low, it will be insufficient when the stream is in flood, so that in flood the lower step in the notch is drowned, and that even the upper step may be in a like

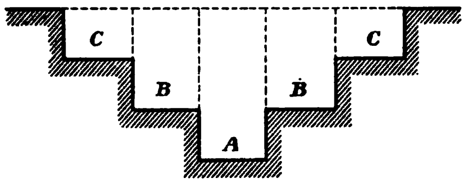


FIG. 91.

condition. It is therefore necessary to have a recorder both above and below the notch, otherwise the time when either notch becomes drowned cannot be ascertained.

Measurement of Large Streams. If the river be large or its gradient slight, it is frequently impossible to institute any reliable notch or series of notches which would give satisfactory results. It may, nevertheless, be possible to construct a dam across the stream provided with a succession of sluices, and to open one or more, at the same time noting the head of water above the sluice as compared with that below it, and keeping a record of the area opened.

It will sometimes happen that it is impracticable to establish any gauge-weir. In such cases one of the following methods must be adopted.

(*a*) Measurement of the average velocity, by timing the passage of floats over a measured length of approximately uniform section.

(*b*) Measurement of the average velocity by means of a current meter.

(*c*) Approximate measurements by means of surface velocities obtained by the use of shallow floats.

Whichever method be adopted, the stream itself must be carefully surveyed, it being of the first importance that the length of the river selected should be nearly straight, that it should be of nearly equal depth and sectional area throughout, and that the stream lines should be regular and unaffected by any curve or bend in the river bed.

If the river be much obstructed by weeds or vegetation, whether up to, or below the surface, the use of deep floats or of the current meter becomes difficult and tedious, if not absolutely impossible.

If a really suitable section has been selected, the greatest velocity of the stream should be found, when there is no wind, in or near to the centre of the river and a little below the surface. This will decrease, with the depth of the stream, towards the banks, until at the bank it will be practically nil, unless the latter be very steep and smooth.

The velocity is at its maximum a little below the surface of the stream (provided there is no wind blowing down-stream), and decreases with increased depth until at the bed of the stream it is actually, or approaches very closely to nil.

With a stream of moderate depth and even cross sectional slope, the velocities

should decrease towards the banks in the form of a parabola, and the vertical velocities will also approach the same form.

Types of Floats.

Figs. 92 and 93 show what are known as "sub-surface" floats. They consist of two parts, the lower part being designed to present a considerable surface to the water. It is very slightly denser than water. The upper part is made lighter than water and as small as possible, so as just to float the whole and be visible from the banks, while presenting as little surface as possible to the water. The two parts are connected by a fine cord, and the whole is designed to give the velocity at the depth of the lower part.

Fig. 94 shows a 'twin float,' in which the two parts are equal, designed to give the mean between the surface velocity, and that at the depth of the lower sphere.

For surface velocities, almost anything which will float with the greater part of its surface immersed (for example, a securely corked bottle ballasted so as just

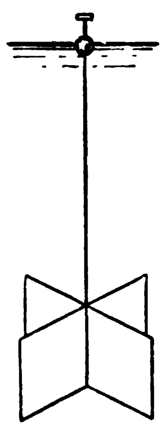


FIG. 92

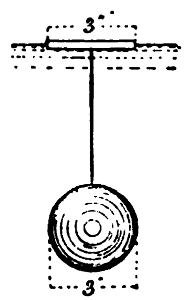


FIG. 93.

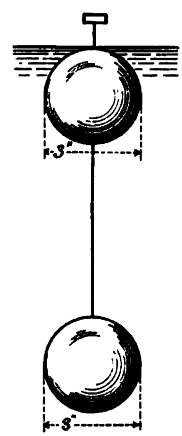


FIG. 94.

to float vertically) will answer, or rods of uniform diameter ballasted to float vertically may also be used to find the average velocity for the whole depth of the rod, and are known as 'rod' floats.

Current Meters or Floats.

The volume of water flowing down a stream may be ascertained with considerable accuracy by the use of a current meter, or by means of observations made with floats.

The current meter consists of a horizontal spindle to which is attached a fan somewhat similar in form to a screw propeller. By means of a worm and wheel the revolutions of the spindle are recorded on a 'counter' which is divided so as to show the velocity of the current in feet, or a ratio of the same. The time between the ringings of the bell is observed with a 'stop watch,' so that the velocity of the current, in feet per second, can be calculated.

In most machines it is necessary to apply a constant correction which is arrived at by experiment, when the velocities are small.

To indicate the time elapsed, an electrical attachment may be provided, which rings a bell at each 100 revolutions of the fan, and is a great convenience, especially when the depth of the stream is considerable.

To use the meter, a rod is attached to the sleeve supporting the spindle of the fan, and the latter is submerged until the centre of the spindle is some 6 inches under water, and in the centre of the stream. The velocity at this depth is thus obtained and recorded, and similar observations are made at say each foot of depth. A diagram plotted from the velocities will approach the form of a parabola.

If the same process be repeated at distances of say ·5, 10, &c. feet from the centre of the stream, sectional diagrams of similar form may be plotted, and from them, the mean velocity of the stream may be ascertained. These transverse sectional observations may be made at every 5 feet, down a portion of say 50 to 100 feet of stream.

In order that the result may approximate to accuracy, it is necessary that great care should be taken in the selection of the site for the observations.

It is more important that the observations should be made near the middle of a considerable length of straight (where the inequalities in the bottom are few, and the cross sectional area varies little), than that the length observed should be great, although as the distance through which the observations are made increases, errors in time, observation, in accidental undetected retardation of the fan, and variations in velocity due to change in the direction of the stream lines, are minimised.

The same results may be obtained with float observations (with greater rapidity), if the section of the stream be free from inequalities of bottom, and from weeds and vegetable growths, so that a simple cylindrical rod (which will extend from the surface almost to the bottom) can be used.

Such a rod may be constructed in lengths screwed together, or with a telescopic length, so that it may be readily adapted to different depths of cross section. If provided with a small box or cage at the bottom into which shot may be placed, the top of the rod may be brought just to the level of the surface, and the time taken in floating a given distance in a given time being observed, the mean velocity of the stream along the course traversed by the float, is obtained.

Float observations should be made when there is little wind, for a slight breeze either up or down stream will materially alter the time occupied by the float in traversing a given distance.

If care be taken in selecting the section of the stream to be used for the purpose of observation, very fair results (approaching in accuracy to the more elaborate systems shortly described above), may be obtained by a succession of tests of the surface velocity of the water made with ball-shaped floats submerged to the extent of two-thirds of their diameter (in default of anything better oranges will answer the purpose). The use of surface floats should obviously be limited to still weather.

The distance of the float from either bank, or from the centre of the stream, where it crosses each section of the length considered, should be recorded, as well as the time occupied in passing from end to end of the total length. The velocity

of the surface stream lines can thus be plotted, and will form a curve of greater or less regularity according to whether the cross section of the stream be regular or otherwise. The area will approximate to that of a parabola with the greatest observed velocity at the centre of the stream.

Any convenient number of parabolas may be plotted on each cross section, from the surface curve to that at the bottom of the stream, so that from the whole of them the mean velocity for the cross section will be obtained.

From these observations it is found that on a carefully selected section, when

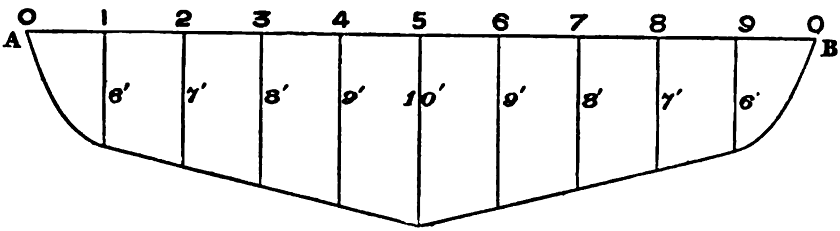


FIG. 95.

the maximum surface velocity is at or near the centre of the stream, a single set of observations for arriving at the central surface velocity, may be used. A parabola may be drawn through the line so plotted from either bank, and the mean surface velocity will be determined from this curve.

Example. The system described above may be much shortened in practice, and as the method to be adopted must be arrived at to suit the circumstances, by each observer, it only remains to give a simple illustration by way of example.

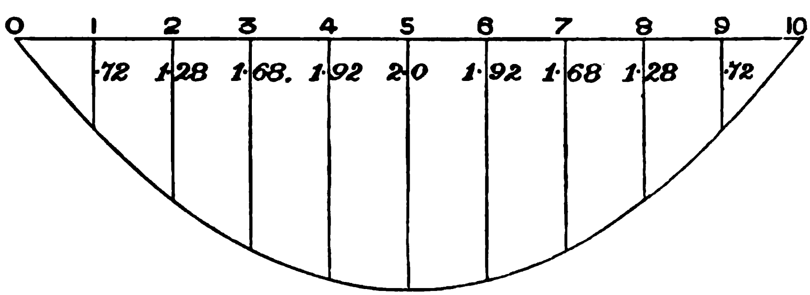


FIG. 96.

Let A B, fig. 95, represent the average cross section of the stream, divided into ten equal distances (each = 5 feet say) by the vertical lines 1, 2, 3, etc.

Let the depths be 0, 6, 7, 8, 9, 10 in feet.

Let the surface velocity at the centre section be 2 feet per second and the sectional area 350 square feet.

The average surface velocity is shown in plan by the parabolic curve below (*vide* fig. 96) and will amount to $2 \times \frac{2}{3}$ or 1·393 feet per second, nearly.

Similarly, if the velocity curve at each vertical be taken as a parabola with its

axis in the surface, the mean velocity in each vertical will be $\frac{2}{3}$ of the surface velocity at that vertical.

Thus starting from the centre, the mean velocities in the verticals will be

(1) $2 \times \frac{2}{3} = 1 \cdot 333$ feet per sec.	(2) $1 \cdot 92 \times \frac{2}{3} = 1 \cdot 28$ feet per sec.
(3) $1 \cdot 68 \times \frac{2}{3} = 1 \cdot 12$ „	(4) $1 \cdot 28 \times \frac{2}{3} = \cdot 853$ „
(5) $\cdot 72 \times \frac{2}{3} = \cdot 48$ „	(6) 0.

The mean velocity at the middle section will be

$$\frac{1 \cdot 333 + 1 \cdot 28}{2} = 1 \cdot 306 \text{ ft.}$$

and the volume, $1 \cdot 306 \times \text{area} = 1 \cdot 306 \times 47 \cdot 5 = 63 \cdot 46$ cubic feet.

The mean velocity at the second section will be

$$\frac{1 \cdot 28 + 1 \cdot 12}{2} = 1 \cdot 2 \text{ ft.}$$

and the volume, $1 \cdot 2 \times \text{area} = 1 \cdot 2 \times 42 \cdot 5 = 51 \cdot 00$ cubic feet.

The mean velocity at the third section will be

$$\frac{1 \cdot 12 + 0 \cdot 853}{2} = 0 \cdot 986 \text{ ft.}$$

and the volume, $0 \cdot 986 \times \text{area} = 0 \cdot 986 \times 37 \cdot 5 = 36 \cdot 97$ cubic feet.

The mean velocity at the fourth section will be

$$\frac{0 \cdot 853 + 0 \cdot 48}{2} = 0 \cdot 667 \text{ ft.}$$

and the volume, $0 \cdot 667 \times 32 \cdot 5 = 21 \cdot 67$ cubic feet.

The mean velocity at the fifth section will be

$$\frac{0 \cdot 48}{2} = 0 \cdot 24 \text{ ft.}$$

and the volume $0 \cdot 240$, $0 \cdot 24 \times \text{area} = 0 \cdot 24 \times 15 \cdot 0 = 3 \cdot 75$ cubic feet.

Total volume of one side $= 176 \cdot 85$ cubic feet.

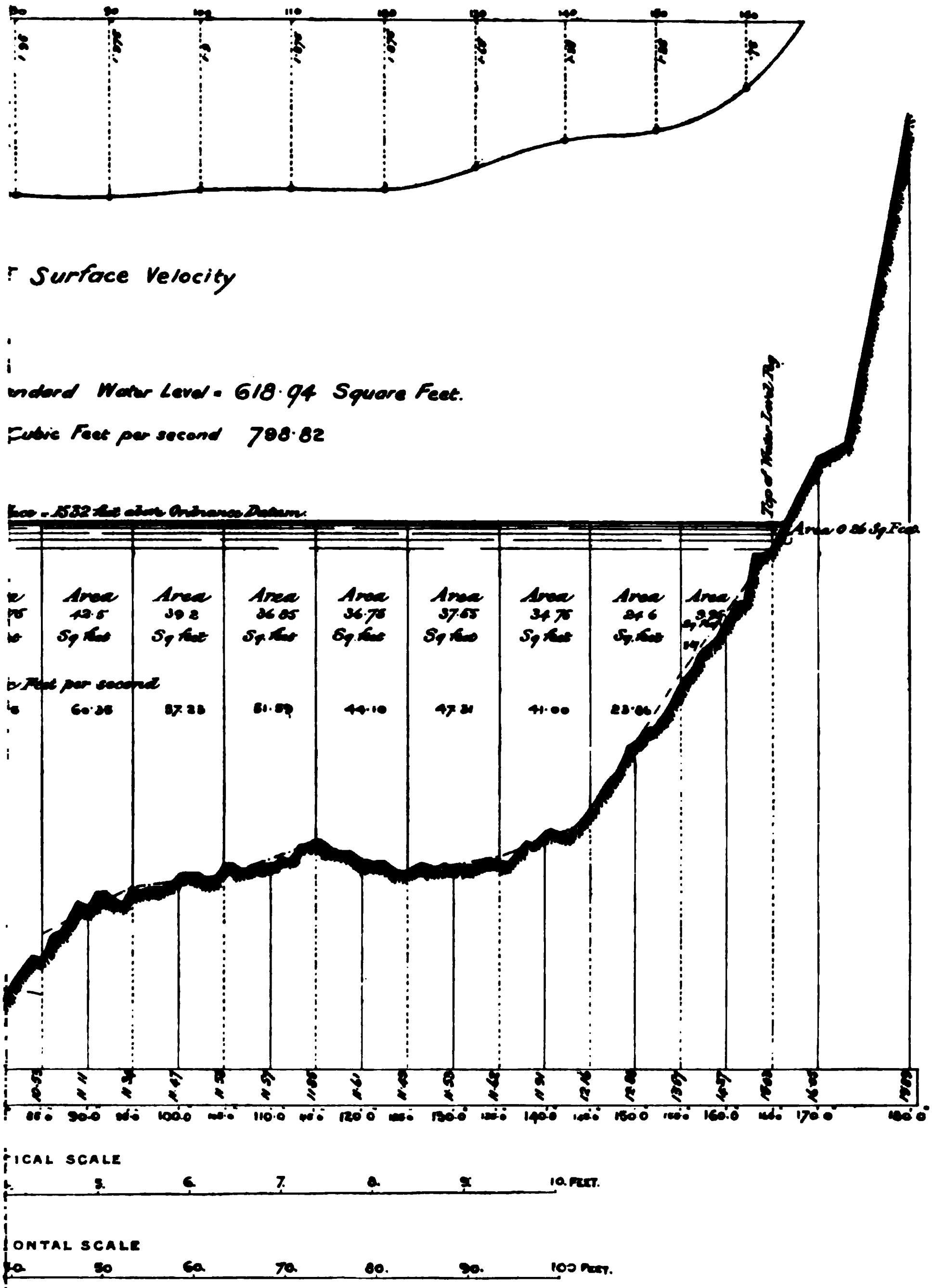
The volume of water passing the section will be $176 \cdot 85$ cubic feet $\times 2 = 353 \cdot 7$ cubic feet per second.

If the maximum central surface velocity be multiplied by $0 \cdot 691 \times 0 \cdot 691$, the flow per second would be 350 cubic feet $\times 2 \times 0 \cdot 691 \times 0 \cdot 691 = 334 \cdot 2$ cubic feet per second, or $94 \cdot 5$ per cent. of $353 \cdot 7$ cubic feet per second, the figure arrived at by the longer method.

If a current meter be used, it should invariably be checked both before, and after the trials, by passing it at a known ratio of speed (approximating to the maximum and minimum flow of the river) through still water, or by passing it at a known rate of speed both with and against the slowest current obtainable.

The meter must be examined frequently to make certain that weeds, etc. have not collected on the spindle, and retarded the rotation of the fan.

Any abnormally slow readings should be discarded. The meter should be



97 and 98.

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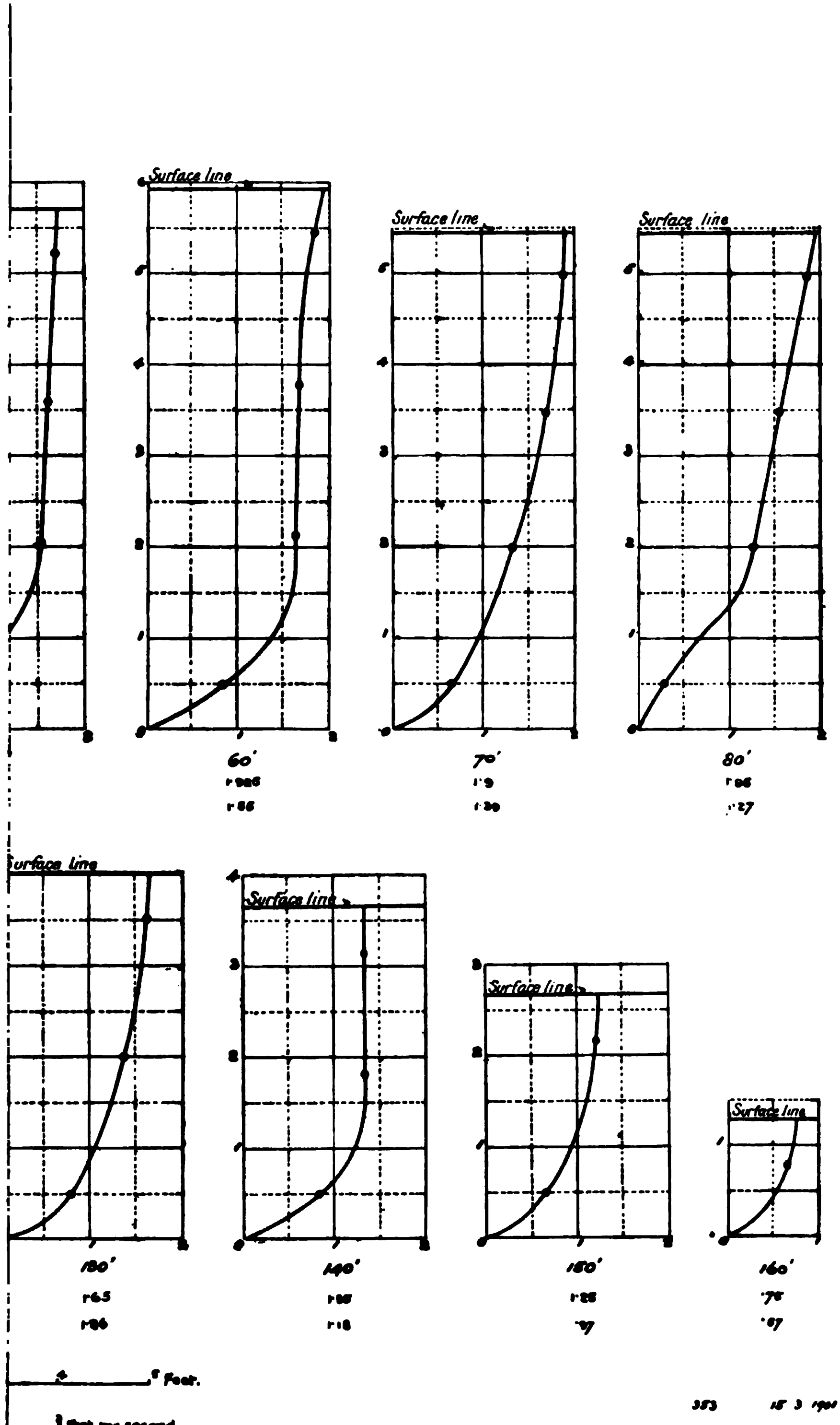

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supported in each case at a known depth below the surface of the water. The method of doing this must, however, depend on the construction of the meter used.

Readings cannot of course be taken exactly at the surface of the stream, for the fan would not be submerged, and here again the surface is affected by the direction and strength of the wind. The *highest* reading should therefore represent the velocity at 6 inches below the surface.

The *lowest* reading cannot be taken at the bottom of the stream, for the fan could not turn, and it is seldom possible to approach the bottom very closely owing to the pressure of weeds and vegetable growth, as well as the unevenness of the bed.

If the readings be taken at 1 foot apart in depth or thereabouts, these numbers should be ample.

A suitable position for gauging stations, with sections of the river bed, and the velocities in diagrammatic form, are shown in figures 97, 98 and 99. The dimensions were taken from observations actually made in practice.

CHAPTER XVI.

SURVEYS OF COAST LINES, HARBOURS, ROADSTEADS, RIVERS, ETC.

Object of Chapter.

THE object of the present chapter is to explain the special methods, artifices, and appliances, used in preparing engineering plans of harbours, rivers, estuaries, and roadsteads.

It is frequently necessary to delineate the foreshore of coast lines, harbours, or estuaries, as defined by high or low-water mark, as well as to survey in the spot soundings or submarine contour lines (as determined by soundings of equal depth). An accurate knowledge of the bottom formation is very necessary before designs can be prepared for works, such as sewer outlets, wharf or bridge piers, dock wall foundations, etc. The method of procedure, when preparing such plans, will now be discussed.

Datum Plane, Soundings, and Levels.

In such plans, whether of ground above or below water, the levels given must be referred to some one datum plane, preferably, at a height, below the level of the bottom in the deepest water that occurs in the plan, and certainly, below the level of the lowest constructions under consideration, such as the foundations of dock walls, wharves, etc.

This avoids the necessity for distinguishing between levels *above* and *below* datum. For example, in the survey of the foreshore of Bombay Harbour, the levels both above and below low-water mark, were referred to what is known as General Delisle's datum, which is a plane 100 feet below a mark on a step of the Town Hall. Mean sea level stands at 83·30 feet above this datum. Extreme low-water mark is about 72·00 feet and extreme high-water mark about 91·00 feet above this datum plane.

All the foundations of dock walls, even of the bottom of the harbour in the fair-way, have therefore in this case, positive levels.

In this respect, an engineering plan differs materially from a nautical chart, the object of which is to show the least depth of water that is likely to be found. In charts, soundings or depths are shown, and these are referred to *extreme* low-water mark. This is not a fixed level plane, but varies from day to day, as well as at different parts of the coast, according to the range of the tide. Thus, in some parts of the British coasts low-water mark may be only four or five feet below mean sea level, whilst at others it may be fifteen, twenty, or more.

Marine surveyors determine a low-water level for each chart. Usually, this is fixed for reference, by stating that it is so much *above* or *below* a certain permanent mark, such as a bolt in a sea wall, or on the sill of some dock, etc.

By ascertaining the level of the mark, soundings may be converted into levels, by deducting them from the level representing low-water mark.

Tide-Gauge. The first step towards making a marine or river survey is to establish a tide-gauge or river-gauge, at or near the scene of operations. If a pier or wharf be available, then the gauge can be fixed at a place where there are always a few feet of water.

Self-recording tide-gauges are made, but their cost is great, and, on account of the clockwork, they require a firm foundation. They take more looking after than would be supposed. There is always a chance of the clock stopping, or of the string to which the float is attached breaking, thus interrupting the record, and perhaps invalidating a whole day's work. On the whole, the plain float and staff, read hourly or half-hourly by a suitable man, is the most practical plan, unless the operations are so extensive as to justify the erection of a permanent self-recording gauge, and the employment of a competent person to look after it. Even if such a gauge be provided it would be well to have a simple float and staff in addition, as a stand-by in case of accident.

Having decided on the datum plane of the survey, such being at a certain number of feet below the level of a bench-mark on shore, the next step is to determine the level of the 'zero' of the tide-gauge scale. This is conveniently done by driving a strong staple like the hinge of a gate, fig. 100, into some solid and permanent structure, not far from the tide-gauge, at or about mean sea level so that it may be uncovered twice daily.

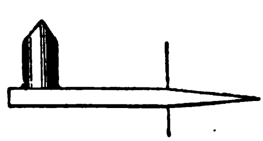


FIG. 100.

The level of the top of the pin is taken in the ordinary way. Then, on a calm day, the gauge is read at the moment at which the surface of the water coincides with the top of the pin. The reading of the gauge *deducted* from the reduced level of the pin, gives the reduced level of the 'zero' of the gauge. The reduced level of the water surface at any moment may be obtained by *adding* the gauge-reading to the zero level. The staple also serves as a point of reference whereby the 'zero' of the gauge may be checked from time to time, to see whether it has shifted on account of the float having become water-logged, or from any other cause.

If there be no solid structure, such as a pier or wharf extending below low-water mark, to which a tide-gauge can be attached, then a series of gauges may be used. Stout stakes may be driven firmly home into the ground, to which graduated scales are to be attached, as shown in fig. 101. These scales must be carefully adjusted with a level and staff, as the tide recedes, so that their divisions coincide with reduced levels, and so that the bottom division of each stake coincides with the top division of the next lower, and so on.

If there be much boat traffic it will be prudent to make the stakes numerous and short, for if long they would be liable to be knocked down, or might cause damage to craft.

Survey of Foreshore. Having established the tide-gauge, a survey is made of the foreshore or coast-line, or of the banks of a river, by the methods already described. Such detail as may be desirable

will be delineated, such as sea walls, wharves, detached buildings, and the like.

As far as the marine survey is concerned, it is merely necessary to fix numerous points along the foreshore. If the survey is to extend out to a considerable distance seaward, it is desirable to fix by intersections, the position of conspicuous objects such as steeples, chimneys, flagstaffs, lighthouses, conspicuous trees, and the like. Prominent rocks may be cut in and made conspicuous by a coating of whitewash, or piles of stones may be erected and whitewashed.

Sections of Foreshore. The next step will be to take numerous sections of the foreshore down to low-water mark. This is done with the ordinary level and staff. The section lines may be set out with a theodolite or pocket-sextant, making some definite angle with the terrestrial survey lines, and connected to the survey points by chain measurements.

The distances from the survey line to the successive positions of the staff may be measured with the ordinary chain or steel band, or they may be determined tacheometrically or by subtense measurements, or by triangulation.

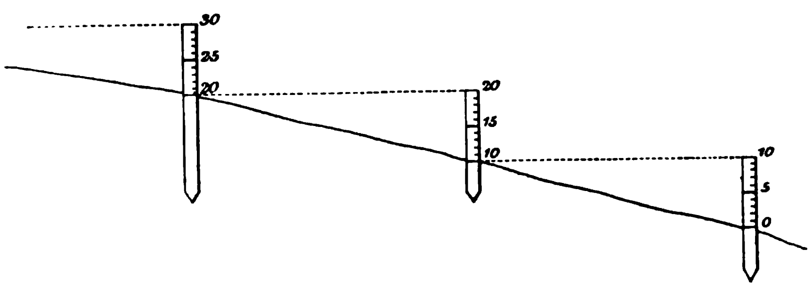


FIG. 101.

Levelling Operations. Levelling operations should be extended down to low-water mark, and even below, to the extent that the staff-holder and surveyor can wade in.

Levelling is more accurate than sounding, and more easily reduced and plotted. Advantage should be taken of low tides, to level as far as possible.

It often happens that very low tides occur at night. Levelling may be easily conducted at night by providing a means of illuminating the field of the telescope, so as to render the cross-hairs visible. This may be done by soldering to the dew-cap of the telescope a mirror of silvered brass or clean tin-plate making an angle of 45° with the axis of the telescope (*vide* fig. 102). An elliptical hole is cut in this mirror, the shorter axis of the ellipse being about three-fourths of the diameter of the object glass. The annular mirror reflects the light of a lantern or torch and renders the wires visible, whilst the staff, illuminated by a torch, is visible through the hole in the mirror. A slip of white drawing-paper about $\frac{3}{16}$ inch wide, tied to the dew-cap, and bent 45° across the object glass answers very well.

Sounding Operations. Sounding is conducted on similar principles to levelling. The surface of the water corresponds to the line of sight of the level The water level is determined by the tide-gauge, and, by deducting from its readings the synchronous depths of water, the spot levels of the bottom are obtained.

An observer is stationed at the tide-gauge, and records its readings half hourly, or more frequently near high and low water. The time of taking each sounding is noted also. The water level at the moment of sounding can be thus ascertained from the final tide-gauge record, by interpolation. If the tidal range is great, the tide-gauge readings may be plotted as a curve and the level scaled off at any required time.

In the case of open coasts, bays, or harbours, a single tide-gauge will serve for an area of several square miles. The level will be practically the same at all points at the same instant of time. Not so, however, in the case of rivers, creeks, or estuaries, where the water level at the same instant will vary very materially at points not far apart. Except at the moments of 'slack water' or the 'turn of

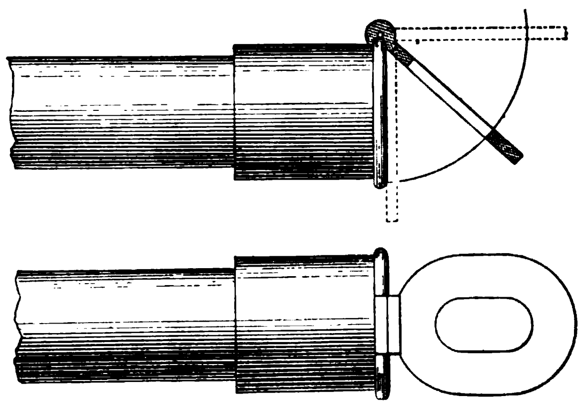


FIG. 102.

the tidal current' the water will be in motion, and consequently its surface will not be level. The duration of 'slack water' is short, so that if the surveyor limits his work to this period, very little can be done each day. In such cases, therefore, it is necessary to have numerous subsidiary tide-gauges, and to use the readings of each one separately for the reduction of the soundings taken in its immediate vicinity.

If the object of the survey be the preparation of a project for the improvement of the navigation, numerous tide-gauge stations and continuous observations at each, are essential to a proper knowledge of the tidal action, apart from the question of soundings.

In some cases, the difference of level in even a short reach of a river may vary so considerably, that the actual level of the water on the spot should be taken by level and staff immediately *before* and *after* taking a line of soundings, the water level for each sounding being interpolated. An assistant, provided with a level and staff, might take the level of the water at frequent intervals in the vicinity of

the sounding boat. In any case, actual levelling should be used as far as practicable.

Staff for Soundings.

For depths not exceeding 15 feet, a graduated staff is preferable to a sounding line. It is well to weight the bottom with lead to such an extent as to cause the rod to float upright. If the bottom be soft, the rod should have a flat foot. Longer rods may however be used if the lines of soundings be taken, as hereafter described, parallel to the stream line, the boat being allowed to drift with the stream.

Steel Bands or Wire as Lead-Lines.

For greater depths, a steel band or wire with a 7 lb. weight attached to it may be used, or a common linen tape will serve if its length be checked. In deep water, and especially in a strong run of tide, a steel pianoforte-wire graduated by attaching small brass labels to it can be used with advantage, as it offers less resistance to the water than a flat tape, and so can be more easily brought into a perpendicular position.

The ordinary hempen lead-line stretches so much, and requires such frequent correcting, that it is unsuitable for accurate work. In the offing, where an error of a foot is unimportant, it may be conveniently employed. It should be graduated to single feet, and not to fathoms only.

Field Book.

The field book for recording soundings should have the following principal headings:—

Distances.	Time. h.m.	Soundings.	Tide-Gauge.	Reduced Levels.	Remarks.

The tide-gauge readings are filled in on the completion of the day's work.

The object to be aimed at, is to cover the bottom of the submerged area with spot levels. In order to ensure regularity, the soundings are usually taken at equal intervals of from 50 to 100 feet in straight lines parallel to each other. Where rocks are suspected to exist, the intervals should be reduced, to prevent a danger from escaping detection.

Fixing the Positions of Soundings.

To fix the position of the soundings, the first step is to put in pegs along the survey lines on shore at the desired distance apart. Generally, it will be well that the lines of soundings on an open coast should have some definite bearing, say, north and south, or east and west, or some bearing at right angles to the general trend of the coast. The lines of soundings should be taken at some fixed distance apart, such as 100 feet. The bearings of the survey lines being known, it is only necessary to set out along the survey line successive distances of 100 cosec θ where θ is the angle between the desired bearing of the line of soundings and that of the survey line (*vide* fig. 103). Then, with a theodolite or sextant at '*a*,' points '*b*' and '*c*' can be fixed, so that the line *bc* makes the desired angle with the survey line A B. Flags or signals can then be put up at *b* and *c*. By keeping these signals 'in one' the boat may be steered on the desired line of soundings.

If a river or estuary is to be surveyed, it is desirable that the lines of soundings should be at right angles to the central line of the stream. These lines may be set out at right angles to the survey lines (fig. 104), determining the shape of the bank and checked by measurement along the survey lines on the opposite

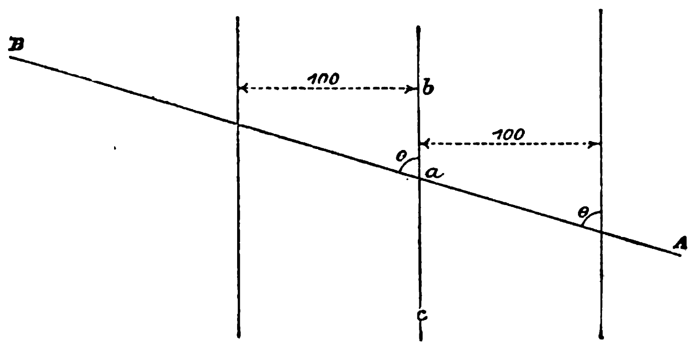


FIG. 103.

bank. Or, if the terrestrial survey has been plotted, giving a general outline of the river bed, the positions of the lines of soundings may be selected on the plan, and the positions of their intersections with the survey lines can be scaled off, and set out on the ground.

Determination of distances of Soundings from the Survey Lines.

To determine the distances of the several soundings from the survey lines, a thin steel wire rope about $\frac{1}{8}$ inch or $\frac{3}{16}$ inch diameter is often useful. This rope is well stretched and graduated to intervals of 50 to 100 feet, either by attaching small brass labels, or by wrapping round it a small coil of brass wire, and securing it with a drop of solder. The knots thus formed,

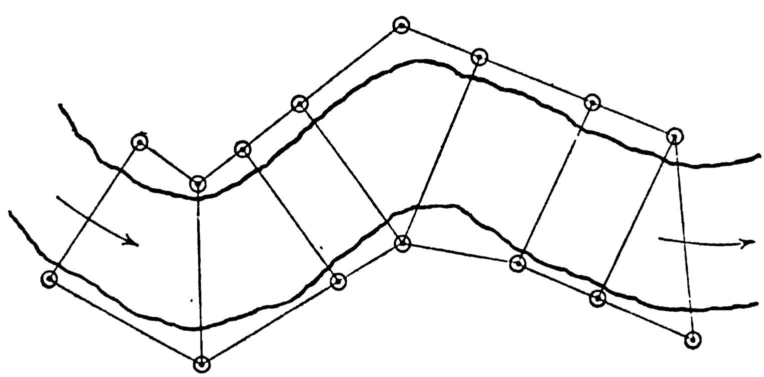


FIG. 104.

may be distinguished by pieces of string, with knots tied to indicate the number of feet. The steel wire rope is strong, and therefore can be strained practically quite straight. A length of a thousand feet may be easily manipulated, so that where there is not much traffic, the rope may be stretched from

bank to bank. In other cases, the outer extremity may be attached to a boat moored in the desired line. The wire rope having been stretched out, it is under-run with a boat and a sounding taken at each knot. The intermediate distances may be measured with a tape or 10-foot rod.

If there be much traffic, it will be impossible to use a rope at right angles to the stream, and also if there be a strong run of tide, the manipulation of the rope is laborious, and much time is lost in running it out, straining it, and mooring and unmooring the boat. So much so, that sounding can in such cases only take place at slack water. As sounding can only be conducted properly when the water is smooth, it is desirable that some method should be adopted which will enable it to go on at all states of the tide.

Measurements in a Stream or Tideway.

In a stream or strong tide-way, the wire rope may be used conveniently, by running it in the direction of current. In this position also it is less liable to be fouled by passing vessels. The following method has been found convenient, in the case of a harbour survey. A piece of log line 2000 feet in length was prepared, and marked at intervals of 100 feet by knotted strings. This line stretched so much that it could not be used as a measure. It was merely used as a means of spacing the soundings between two fixed points, determined as hereafter described.

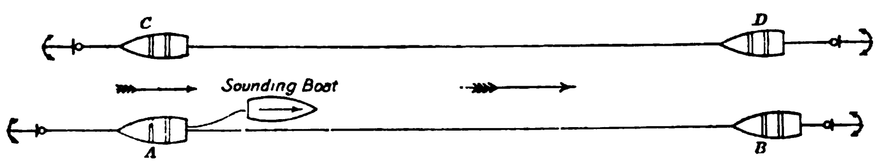


FIG. 105.

A boat was anchored at A, fig. 105, and the sounding-boat proceeded to run out the line to another boat B, anchored down-stream. The line was hove taut, and the position of B was adjusted by paying out cable, until some one knot was brought to the centre of boat. The line was then made fast to the boat B whose position was fixed by angles taken to objects on the shore. Before commencing to sound, another boat or buoy was moored at D about 100 feet from B. The sounding-boat then proceeded to under-run the line, a sounding being taken at each knot. On reaching A its position was also fixed by observation to objects on shore.

The soundings were plotted from the determined positions of A and B, the distance being divided into a number of equal parts, corresponding with the number of soundings. It was rarely found practicable to keep the line quite straight, owing to cross currents, or to the steering of the sounding-boat, and therefore at least one intermediate position was fixed by observation, and the soundings were plotted along a fair curve sketched through the fixed points, spacing them by dividing the line into as many parts as there were knots between the boats.

A boat or a buoy was fixed at C for the new line C D, and the next line was taken in a similar manner.

The position of soundings may be fixed by means of two theodolites or two

plane-tables set up over known points on shore. Or, if the soundings are taken along a definite line indicated by signals or marks ashore, one theodolite or one plane-table would suffice (*vide* fig. 106).

When the boat is brought so that the sounding-rod or line is in line with the two flags or marks on shore, the man in the boat gives a signal and the pole or a small flag in the boat is intersected by the cross wires of the theodolite and a bearing is read off, and ultimately plotted by protractor. With the plane-table, a ray is drawn, intersecting the prolongation of the line joining the two flags on the sheet.

The accuracy of this method depends upon the distance between the two flags and the magnitude of the angle at the vertex, i.e. the boat. If the two flags are near together, the direction of the line of soundings will be imperfectly defined. If the angle at the boat be too acute, the intersection of the rays will be uncertain.

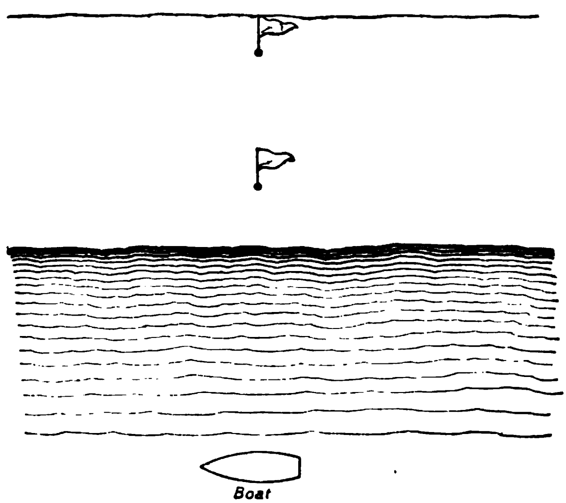


FIG. 106.

The co-operation ot a second observer on shore is inconvenient, and may be dispensed with by the use of the sextant in the boat. The surveyor is then independent. It is evident that if at the moment of sounding, the signals A and B (fig. 107) be exactly in line, or the angle A X C is observed at the boat, the position of the latter is fixed. A second angle A X D would give a check. Indeed, by observing two angles such as B X C and B X D, the position of the boat at X is fixed by the three-point method, independently of the prolongation ot any fixed line such as A B. It often happens that it is impossible to lay out a line A B of sufficient length to be of real use. Setting out the line on shore takes time, so that, on the whole, the three-point method is the most generally useful. Having determined the exact position of several points on shore, signals are put up at once, to permanently mark them. The signals must be conspicuous, far more so than for observing with a theodolite, and care must be taken to suit the colour to the background. Vision is not so distinct with the sextant, as with

the theodolite. In the above manner of fixing soundings, the surveyor is wholly independent of the shore (excepting as regards the determination of the water level), and he can stop afloat as long as the weather is suitable. It is true that without a rope or guiding-lines set out on shore the soundings cannot be quite so neatly spaced as with them, but on the other hand far more can be taken in the day. There should be no difficulty in distributing the soundings over the area in a manner that will delineate the bottom properly, and to this end the surveyor should take with him the means of plotting some of the points, such as the termination of lines of soundings. A second boat with an anchor and cable is useful, to mark the position of termination lines, so that a new line may be taken up.

In deep water, when the highest accuracy is not wanted, it is not necessary to observe at each sounding. The boat may be rowed with a steady stroke, and

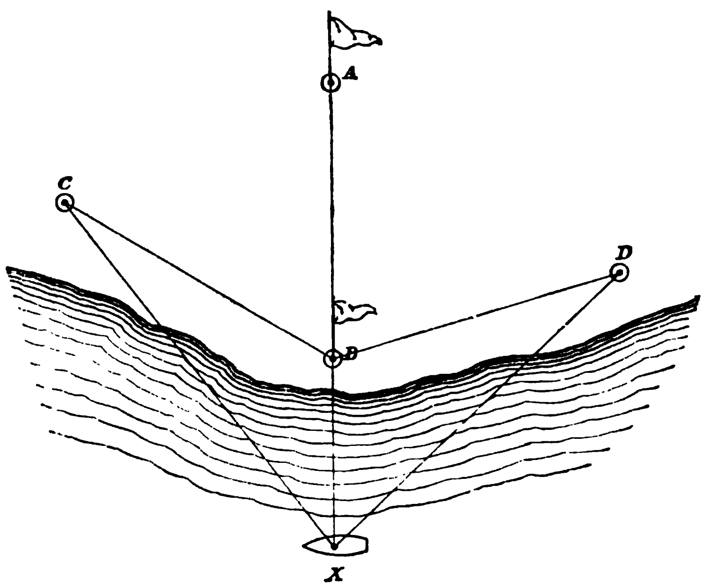


FIG. 107.

a sounding taken at every tenth stroke. At the tenth sounding a heavy lump of iron attached to a rope is dropped to the bottom close to the position of the sounding. The boat is steadied with the oars, and the oars worked so that the rope is nearly vertical. The necessary angles are observed, and the iron weight is taken up and the boat is started again in the desired direction. The boat b'st suited to such work is a light four-oared gig, which can be easily started and stopped.

It is not a bad plan to work parallel to the shore, keeping the boat approximately in some constant depth of water. The soundings will then follow approximately a contour line. Any slight displacement of an individual sounding will not matter so much as if the line were taken at right angles to the shore.

It is rarely necessary to calculate the position of soundings trigonometrically.

The construction for plotting the three-point problem has been given under minor triangulation (Vol. I.).

The following method of protracting positions from angles observed to three points is due to Beaupré. Let A, B, C, fig. 108, be three fixed points. Let X be the position of the boat, and the angles A X C = θ_1 and C X B = θ_2 the angles observed. Add θ_1 to θ_2 and deduct the sum from 180°, calling the remaining angle ρ. Lay off the angles A C D and B C E, each equal to ρ, drawing the lines C D and C E of indefinite length. At B lay off the angle C B F = θ_1, and draw B F cutting the line C E in F, also at A lay off the angle C A G = θ_2, and draw A G cutting the line G D in G, join F A and G B and produce these lines till they intersect in the point X. Then X is the position of the boat.

For plotting bearings, a station-pointer is sometimes convenient (*vide* Chapter I. on Instruments).

It is rather heavy, and if much work has to be done with it the paper is apt to be injured and made dirty.

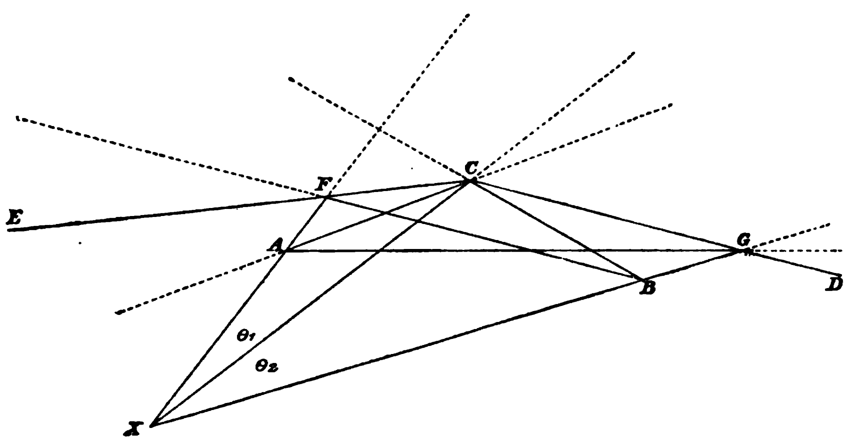


FIG. 108.

A piece of tracing-paper and a good protractor affords a ready means of plotting the bearings observed to three or more points. Lines are ruled on it meeting at a point, and making the observed angles with each other. The paper is applied to the plan and moved about until the lines coincide with the three fixed points, when the intersection of the lines will mark the point sought, which may then be pricked off.

It is always desirable to observe to four points, firstly, for a check, and secondly, because the intersections from three points may be ill-conditioned. If the boat is on the prolongation of some known line, two angles suffice.

The ordinary pocket sextant is hardly a desirable instrument for sounding work. The mirrors are rather small, and it is, therefore, difficult to pick up distant objects quickly. An ordinary 8-inch sextant as used in navigation, is preferable. The Admiralty provides a special sextant for sounding purposes, with 6-inch radius and extra large mirrors.

CHAPTER XVII.

TIDAL PHENOMENA.

Object of Chapter.

THE object of the present chapter is to instruct the Surveyor as to the general character of the phenomena which he may expect to observe in the course of conducting Marine Surveys, and as to the nature of the observations which must be made in order to determine Mean Sea-level, as well as for the determination of the constants required for the construction of tide-tables, giving predictions of the times and heights of high and low water, respectively, day by day. Space will not permit of giving the details of the methods of deducing constants from observations taken, or of the manner of using them for predictions, these operations being too complicated and laborious for the surveyor to undertake.

The writer here begs to acknowledge the valuable assistance which he has received in the preparation of this chapter, from Mr. E. Roberts, of the 'Nautical Almanac' Office, who prepares the tide-tables for Indian and other ports. The tide-tables of Hong-Kong, are based on observations made in the first instance by the writer.

Recurrence of High and Low Water.

It is an observed fact that the surface of the sea rises and falls, in approximate accordance with the apparent movement of the moon,. high water and low water recurring twice daily (approximately), at some fixed interval of time after the moon's transit. Currents also obtain, whose direction and velocity vary in accordance with the same phenomenon. For the present, attention will be confined to the rise and fall ot the surface-levels. Further, it is observed that the *range* of the tide is greater at *new* and *full* moon, than at the *quarters*. This leads to the conclusion that the Sun also has influence on the tides.

The 'range' of a tide is the difference between the water-levels at high and low water respectively. The half-range is called the 'amplitude.'

The greater tides, occurring at 'full' and 'new moon,' are known as 'Spring tides,' the smaller occurring at the 'first' and 'last quarters,' are called 'Neap tides' (*vide* fig. 111). The time-interval between the moon's transit and the instant of high and low water, lengthens and shortens, during a lunar month. This is known as the 'priming' and 'lagging' of the tide.

On Admiralty Charts the words 'High-water at full and change .... hours' occur. This is the hour at which high water occurs at 'full' and 'new moon' (the days, when the moon transits at noon and midnight respectively), so that

the numbers given, are the hours (afternoon and forenoon), of high water, on the days of 'new' and 'full moon' respectively. Since the moon makes a complete circuit, round the earth, in about twenty-eight days, high water will occur roughly, fifty minutes later for each day of the moon's age.

Tidal Establishment of a Place.

The hour of High water 'full' and 'change' is called the 'Establishment of the place.' Suppose that the Establishment of a place is VI. hours. Then at 'new moon' the time of high water will be 6 A.M., followed by a second high water at about 6.25 P.M., on the same day. On the next day, the high water will occur at 6.50 A.M. and 7.15 P.M., and so on till full moon. The 'Establishment' therefore, affords a means of *roughly* determining the time of high water on any given day of the month, by adding fifty minutes, for each day of the moon's age. This method is not accurate, though possibly near enough for purposes of navigation. In the case of important ports, 'tide tables' are published, giving the time of high water day by day.

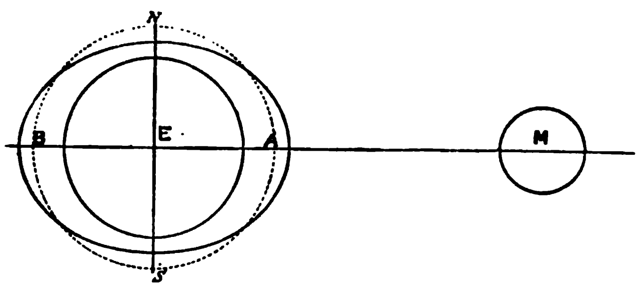


FIG. 109.

Again, the range of the tide does not vary uniformly from springs to neaps, but there are often alternate tides of greater or lesser range. For example, a morning tide may rise to a height of 9 feet, but the following tide in the afternoon may only rise to a height of 5 feet, while on the next day, the morning tide may rise to $8\frac{1}{2}$ feet in each case, above Mean Sea-level. So also for the heights of low water.

Statistical Theory of Tidal Movements.

The theory of 'tidal movements' is extremely complex, though generally speaking, they are undoubtedly due to the attraction of the sun and moon, drawing up the water. An explanation of observed tidal phenomena (as regards their general character), may be obtained, by assuming that the globe is entirely covered with a uniform layer of water, and that the sun and moon move intermittently and not uniformly in their courses, pausing in successive positions for a sufficiently long time, to allow the layer of water to assume the form due to their attractions. This assumption is known as the 'Statical theory of tides,' and was first propounded by Newton.

Let E be the earth and M an attracting body, such as the moon (*vide* fig. 109).

Moon's Effect on Tides.

If there were no attracting body in the neighbourhood of the earth, the water would form a uniform layer round the solid spherical kernel, as shown in the dotted lines. Now, suppose that the moon M came into existence, it would exert an attraction upon the particles of the solid earth, and of the water surrounding it. By the law of gravitation, the attraction of the moon upon any particle of matter, is inversely as the square of the distance between the moon's centre and the particle in question. Consequently, the particles of water at A will be more strongly attracted than those of the solid earth, whose centre of gravity is at E, and these again will be more strongly attracted than those of the remote water particles, at B. The water-surface will therefore assume the form of a spheroid, with its longer axis pointing towards the attracting body. This form will follow the moon in its *apparent* daily course, and the result will be that twice in every lunar day

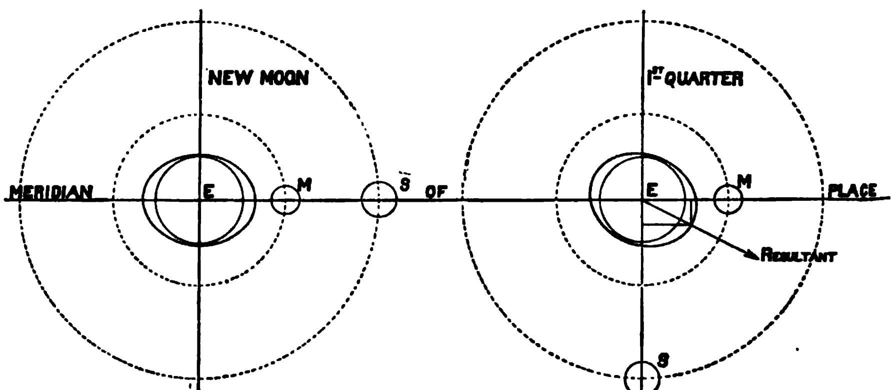


FIG. 110.

(the lunar day of about 24 hours 50 minutes being the average time, in mean solar hours, elapsing between two successive transits of the moon over the meridian of a given place), there will be two 'high waters,' and two 'low waters, at every place. This conclusion is in general accordance with experience, though the instant of high water does not coincide with the moon's transit, but usually occurs somewhat later. This is no doubt due to the fact that the earth is not wholly covered by water. The fact that there are two 'high' and two 'low waters' in a lunar day is thus explained.

Sun's Effect on Tides.

The sun also is an attracting body, and produces similar effects.

Effect of Sun's and Moon's Attractions Combined.

Hence, two high and two low waters in a solar day (24 hours) would take place were the moon absent. Both sun and moon being present, tides would be produced which would be the resultant of their two attractions.

Now the moon in its apparent course lags (on the average about fifty

'Neaps' and 'Spring Tides' Explained.

minutes daily) behind the sun. Consequently, the moon changes its position in the heavens with regard to the sun, making a complete tour of the heavens in a lunar month. At 'new moon,' the sun and moon are in line, and on the same side of the earth (*vide* fig. 110). The two attractions coincide in direction, and a tide equal to the sum of the lunar and solar tides may be anticipated, i.e. a spring tide. So also at full moon (fig. 111), when the moon is directly opposite to the sun. When the moon is at its quarters, or at right angles to the sun, its attraction is exerted on points of the sphere, where that of the sun is producing a depression. The resultant tide is, therefore, the difference of the solar and lunar tides, i.e. a neap tide. This agrees with experience, though the highest tides do not, at all places, coincide with the instants of new and full moon, but often are some days later. The phenomena of 'neaps' and 'springs' are therefore explained.

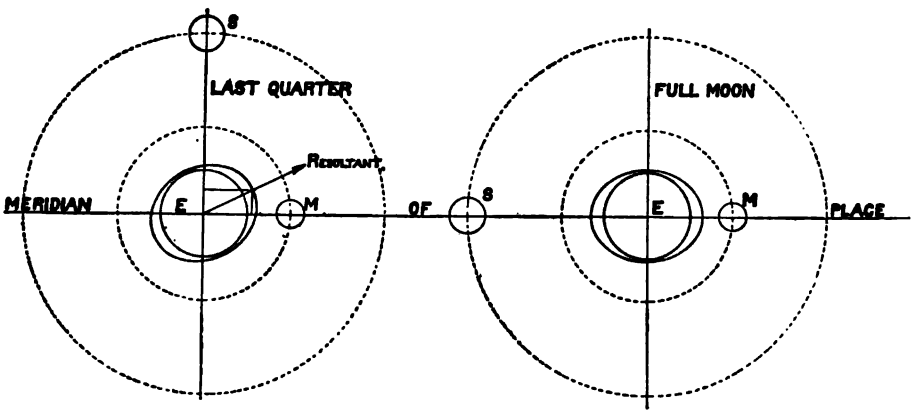


FIG. 111.

The Terms 'Priming' and 'Lagging' Explained.

The phenomena of 'priming' and 'lagging,' may be explained by similar reasoning. The lunar and solar tides combine, and form a resultant ellipsoid, or pair of tides, whose longer axis will be in the line of the resultant of the pair of attracting forces. Taking the transit of the moon as the regulating phenomenon of the tide, because its attracting force is greater than that of the sun, then it is evident from fig. 111 that, at new and full moon, both bodies will be on the meridian of the place of observation at the same instant, and that the resultant attraction will be the sum of the two forces acting in one straight line, and passing through the centres of the three bodies. High water will occur at the instant of the moon's transit, and again 12 lunar hours later. At the first and last quarters, the tidal axis will follow the line of the resultant of the two attractions. At the first quarter, the resultant will be in advance of the moon, so that high water may be expected to occur *before* the moon's culmination, whilst at the last quarter, it will occur *after* her culmination.

Effect of Change of Declination and Right Ascension of Sun and Moon.

Hitherto, we have considered the effects of the sun's and moon's attraction, as though they acted in the plane of the earth's equator. But neither the sun nor the moon are at all times situated in the plane of the equator. The moon changes her declination, that is, the angle which the direction of the moon's centre makes with the plane of the equator (sometimes being as much as $21\frac{1}{3}°$ north, and at others as much south of the equator), the variation taking place in a lunar month. The sun also varies its declination $23\frac{1}{2}°$, from north to south, in the course of a year.

Now it is obvious that when the sun and moon have nearly the same declination, and are on the meridian, the resultant attraction will be nearly equal to the sum of the two attractions. When, however, they have widely different directions, the resultant will be less than the sum, and therefore a

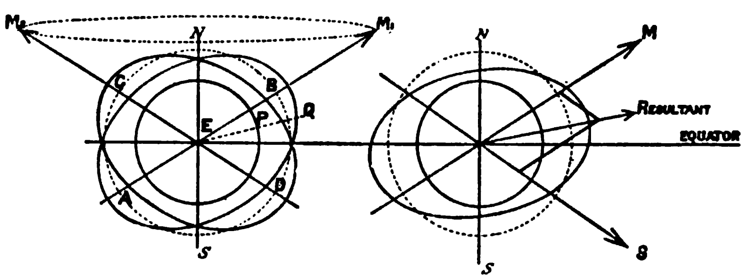


FIG. 112.

variation in tidal ranges may be expected to follow the variations in 'declination,' as well as in 'right ascension,' of both the sun and the moon, the period of the latter being a *lunar month*, and of the former, *a year*. As a matter of fact, it is usually found that extra high and low spring tides occur, at or near the equinoxes (spring and autumn), small ranges obtaining at the solstices (summer and winter).

Diurnal Inequality.

The difference in the height of successive tides is called the 'diurnal inequality,' which in British waters is but small, though very large in other places, such as Aden, where it is sometimes so large, that one tide of the day entirely disappears, being only represented by a *slowing down* in the *rate* of the rising and falling.

Variation in 'declination' serves to account for this phenomenon, and is explained as follows.

Let us consider the moon, in the first instance (*vide* fig. 112).

Let E M be the direction of the moon's attraction, when it passes the meridian of some point P, not in the equator. The longer axis of the watery spheroid will be in the direction A E B M, and the radius of the spheroid, drawn through P will be E P Q. Twelve lunar hours later, the moon will be again in the meridian of P. The longer axis of the spheroid will then be in the direction of D C M_2 and its radius in the direction of E P will be manifestly shorter than

E P Q. The fact of alternate large and small tides, known as the 'diurnal inequality,' is thus explained.

Effect of the Relative Distance of Sun and Moon on Tides.

Now, assuming the attractive forces of the sun and the moon to be respectively *directly* as their masses, and *inversely* as the squares of their distances from the earth, the sun's attraction will be to that of the moon as 1 is to 2·1. The range of spring tides might therefore be expected to bear to that of neaps, a ratio of $2 \cdot 1 + 1 = 3 \cdot 1$ to $2 \cdot 1 - 1 = 1 \cdot 1$, though so great a difference is rarely observed.

Tidal Perturbations.

Lastly, tidal perturbations exist, which are due to the *variations* in the distances of the sun and moon, and to the variations in their rates of apparent motion.

From the above it will be seen that the theory of tides is sufficiently complicated, even on the comparatively simple supposition of an uninterrupted, and completely enveloping ocean. The fact of land and water intervening, renders these problems even more so. Indeed, there is no calculus, by which the time and range of tides and their variations, can be computed, on first principles. It appears from observations, as though the primitive tide-wave were generated in the Antarctic Ocean, moving from east to west. From this wave, a lateral one is thrown off, rolling swiftly northward and up the Atlantic, Pacific and Indian seas,* the line of the crest of the wave becoming more and more convex to the north, as it proceeds. The tide-wave approaches the British Isles from the westward, curling round the north of Scotland, and up the Channel, till two waves meet in the North Sea, one advancing from the south and the other from the north. No doubt the attractions of the sun and moon have some effect on the level of the Atlantic, but the local effect cannot be great, otherwise the tide-wave would not approach Ireland from the west, or in a direction opposite to the movements of the sun and moon.

The range of tides is very variable. Judging from observations taken at small islands rising abruptly from very deep water, such as Mauritius or the West Indian Islands, the maximum range, in mid ocean is but from four to five feet. But when the wave rolls up narrow channels (especially if trumpet-shaped), the range increases greatly, and may be sometimes as great as 80 feet.

Theoretical considerations, based on the assumption of an all-enveloping ocean, merely indicate the kind of phenomena that may be expected at any point on the earth. Experiment shows that the tides do follow, in their variations, the ever-varying relative positions of the sun and moon, in 'right ascension' and 'declination,' but the amount of the influence of each set of conditions, and the time at which it will be felt, can only be determined by actual experiment at each particular spot. In short, theory only indicates what is sought for, but not the extent of the result.

* The wave which represents the tide will be called the 'tide-wave,' because the better expression tidal-wave has been generally used to express an extraordinary wave caused by some convulsion of nature, such as an earthquake.

Anomalous Tides.

The tides of the British waters are unusual in their regularity, and may be predicted with approximate accuracy, by simple processes. This is not the case, however, in many foreign ports, especially in the tropics. The simple rules for prediction, given in Manuals of Navigation, though they suffice for British ports, utterly fail when applied in these cases. Local Authorities, Harbour-masters and others, not unfrequently assume that the tides are utterly irregular and unpredictable, usually attributing the apparent anomalies to a 'breeze somewhere in the offing,' yet numerous cases have occurred within the knowledge of the writer, in which sufficient observations having been made, and properly reduced, apparently anomalous tides have been regularly predicted with great accuracy.

Nature and Extent of Tidal Observations.

To obtain reliable information, as to tidal phenomena, it is necessary to take hourly readings of the sea's level, above some Datum-plane. These must be continued night and day for 369 days 3 hours. A mere record of the *times* and *heights* of high and low water will not suffice, indeed such observations will most probably give very erroneous results. Even mean sea-level cannot be obtained accurately in this manner. Neglect to take continuous observations, is the principal cause of tidal phenomena being pronounced to be anomalous. Had continuous observations been taken and plotted as a curve in such cases, it would have been obvious that the rise and fall of the tide followed a regular periodic law.

An approximation to mean sea-level may be obtained by observations extending over a lunar month, but absolute accuracy can only be arrived at by observations extending over the period stated above.

Assumptions on which Tides are Investigated and Predicted.

Guided by the general principles which have been briefly described, and having to hand the results of *continuous* observations, extending over 369 days 3 hours (25 lunations or lunar fortnights), tides may be predicted with any degree of accuracy. To do so, the following fiction is used. The actual tide curve, as obtained by plotting hourly observations (taken during the period named above), is assumed to be the resultant of a number of tide-waves superimposed, each having different 'periods' between successive high waters, and different 'ranges,' that is to say differences of level between high and low water. The range of each *individual* tide is assumed to be constant. The *periods*, or times elapsing between the successive high water of each component tide, are also assumed to be constant, and to bear some fixed proportion to the real mean motions of the real sun and moon, such as a solar day, a lunar day, half a solar and half a lunar day, a fortnight, a year, and so on.

The rise and fall of each tide is assumed to obey the law of a harmonic curve or oscillation, a term which will be explained later on.

It is capable of demonstration, that the elevations at opposite sides of the earth produced by the attraction of the moon, would be produced to the same degree, and to the same extent, if the moon were cleft into two equal parts, placed equidistant from the earth and opposite to each other. The moon and

ante-moon would transit successively at intervals of 12 lunar hours, producing tides at these intervals. So also for the sun. Then, to produce the effect of diurnal inequality, a further moon and sun are imagined, passing the meridian at intervals of 24 hours, and so on.

A number of imaginary stars are assumed, each producing a high tide, immediately under it, but *not* opposite it. The period of these stars, or their times of apparent revolution, are assumed to be multiples or submultiples of the real mean motions of the sun and moon. Imaginary stars are also assumed, to take account of perturbations produced by the variations of the *rate* of movement of the moon and sun, and of their distances from the earth, as well as by the retardation of the tide-wave in flowing up an estuary. The problem is then—

(*a*) To find by observation the range of the tide, produced by each imaginary star.

(*b*) To find positions of the heavens, occupied by each of these imaginary stars, at some one instant of time, such as midnight on January 1, with regard to the position of the mean sun. This known, their position at any other time may be predicted, and the time of high water at any place which each produces, may be ascertained.

Harmonic Curves or Oscillations (above referred to).

If hourly observations were made of the sea-level above some datum plane, and then plotted vertically from a base-line, the ordinates representing 'height,' and the abscissæ 'time,' a line, sketched through the successive points, would form an undulatory curve, having the character of a compound 'harmonic curve.' A simple 'harmonic curve' can be traced mechanically by the following device.

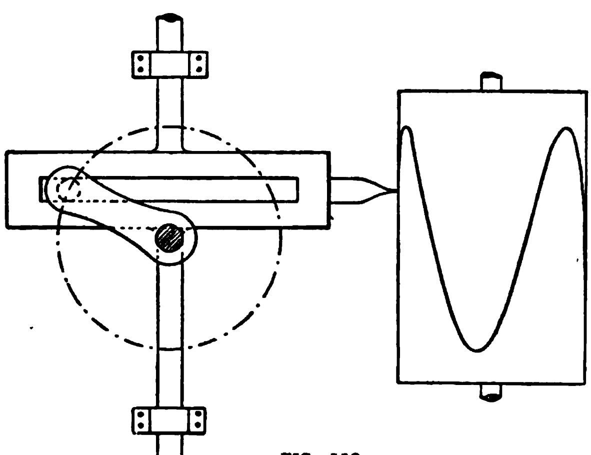


FIG. 113.

Let the pin of a crank engaging in a slotted link, be guided in a vertical direction, so that the link can move up and down in a vertical direction only, whilst the crank revolves. In short, an arrangement sometimes used for donkey

pumps (*vide* fig. 113.) Let the crank revolve uniformly. A 'harmonic motion' will be imparted to the link.

Now let a pencil be attached to the link, marking upon a vertical drum, and let the drum be caused to revolve at a rate proportional to the rate of revolution of the crank, by connecting it to the crank shaft by gearing, so that when the crank makes one, two, or more revolutions, the drum makes one revolution.

Then the pencil will trace a simple 'harmonic curve' on the drum.

Graphic Construction of Harmonic Curves.

A harmonic curve may be constructed graphically, as follows :—

Describe a circle, and divide its circumference into a number of equal parts, say twelve. Through the centre of the circle draw a horizontal line, and along it lay off a number of equal parts, numbering them 0, 1, 2, 3, ... 12. Mark also the division points of the generating circle 1, 2, 3, ... 12. Then, through the divisions of the horizontal line draw verticals, and intersect them by horizontal lines, drawn through the corresponding division points of the generating circle. Sketch a fair curve through the points of intersection. This curve will be a simple harmonic curve (*vide* fig. 114).

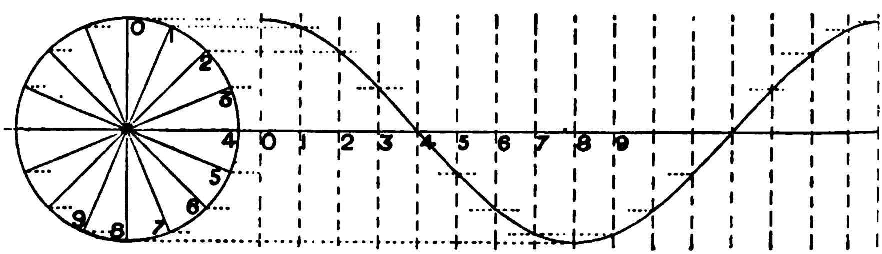


FIG. 114.

Application of Construction to Tidal Curves.

Now, let the diameter of the generating circle represent, to some suitable scale, the range of a tide, that is to say the difference of level between high and low water. Let the circumference of the generating circle represent (like a clock dial) the 'period' of the tide, that is the time-interval between the instants of two successive high waters. For example, if it be desired to describe a lunar semi-diurnal tide-curve, the *whole* circumference will represent *half a lunar day* (12 h. 25 m. *solar* time). The twelve arcs will each represent a lunar hour, about sixty-four solar minutes. With the same scale used to lay down the range of the tide, lay off *downwards*, the height of mean sea-level above the datum or zero plane. Draw a line through this point parallel to the line of mean sea level, and produce the ordinates of the curve to cut it. The ordinates of the curve, or heights of intersecting points will represent the height of the waterabove datum, at successive *lunar* hours, counted from the instant of high water. If the horizontal time-intervals were each subdivided into sixty equal parts, then the heights of the water could be scaled off for intermediate *lunar* minutes, but what is usually

required, is the height at *mean solar* hours. To obtain this information, it is merely necessary to regraduate the horizontal line, with equal and smaller intervals in the ratio of a lunar to a mean solar hour, i.e. in the ratio of 62 to 60 (approximately). Suppose for example, that the horizontal scale of time is one inch to one hour of *mean solar* time, then, for laying off the lunar semi-diurnal tide-curve, the horizontal time intervals would be each $\frac{62}{60}$ of an inch. Having plotted the curve, with abcissæ-intervals of this length, the height of the water level at any solar hour-interval (counting from time of high water), could be scaled off by drawing ordinates at intervals of $\frac{62}{60}$ of an inch, and so also for the intermediate minutes. If the mean solar hour of high water be known, then the height of the tide at any hour, and minute of the day, could be ascertained, merely by shifting the graduations of the time scale to the right or left, so as to make the instant of high water, as determined by observation, coincide with the high water as given by the curve.

A solar semi-diurnal tide can be projected, in similar manner. In this case, the twelve divisions of the generating circle would represent solar hours, so also the equal divisions along the base line (*vide* fig. 115).

Compounding Harmonic Curves.

Suppose now that a lunar semi-diurnal tide-curve were traced in the manner described (not merely for a single lunar day, but repeated day after day, for a considerable period, say twenty-eight days), and that the horizontal hour-intervals were each $\frac{62}{60}$ of an inch (or 1 inch = 1 *solar* minute), also that the scale for heights were 1 in. = 10 ft., then, to plot a semi-diurnal solar tide-curve (comparable with the lunar semi-diurnal tide-curve, just described), the same scale must be used for the diameter of the generating circle, representing the range of the semi-diurnal solar tide, but the horizontal hour graduation would be $\frac{62}{60}$ of an inch, thus representing *mean solar* hours. The zero-point of the solar semi-diurnal tide-curve should also be shifted, as described in the case of the semi-diurnal lunar tide-curve, so that the summits of the curve correspond with the observed time of semi-diurnal solar high water, and that the height of the water level may be scaled off at the various mean solar or ordinary hours of the day. Inasmuch as the solar semi-diurnal high water will occur at the same hours, day after day, there will be no necessity for repeating this curve for a series of days, as in the case of the lunar tide curve. Having thus plotted the two curves, to the same scales of time and height, they may be combined in the following manner.

Draw a horizontal line, to represent mean sea level, and divide it into equal parts each representing one *mean solar* hour, twenty-four spaces representing an ordinary day. Number the points of division 1, 2, 3, 4, ... 12 noon—1, 2, 3, 4, ... 12 midnight, or from 24 midnight to 24. Draw ordinates through the hour-points, *above* and *below* mean sea level. Continue the graduations for a number of days, at least fourteen. Next, scale off the ordinates of the lunar semi-diurnal curve, at each successive *solar* hour-division, measuring them,

not from the Datum line but from the line of mean sea-level. Now, scale off the ordinates of the solar semi-diurnal curve at hour-intervals, measuring them, also, from the line of mean sea-level. When the hourly levels are both *above* or both

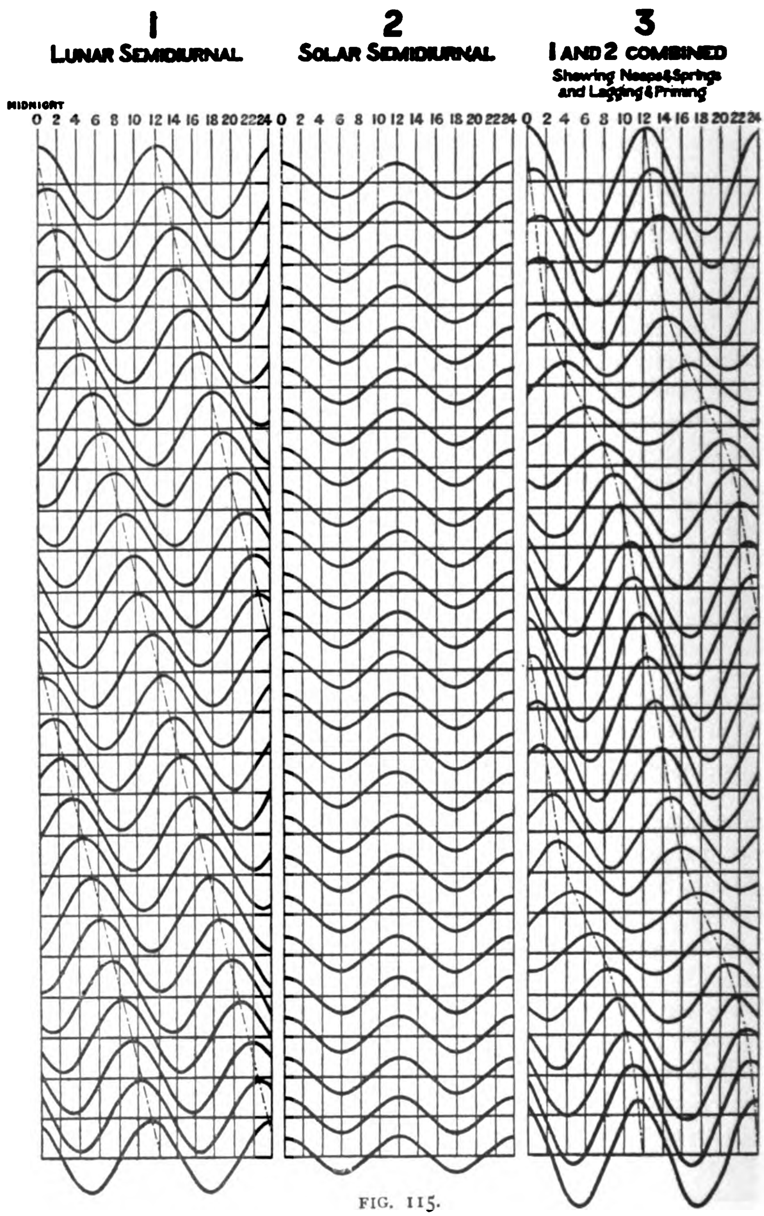


FIG. 115.

below mean sea-level, add them together, and plot the sum *above* or *below* the new line of mean sea-level, at the appropriate hours. When one is *above* and one *below*, take the difference, and plot it *above* or *below* the mean sea-level line.

according to the direction of the greatest ordinate. The addition or deduction may be performed, graphically, by means of a pair of dividers.

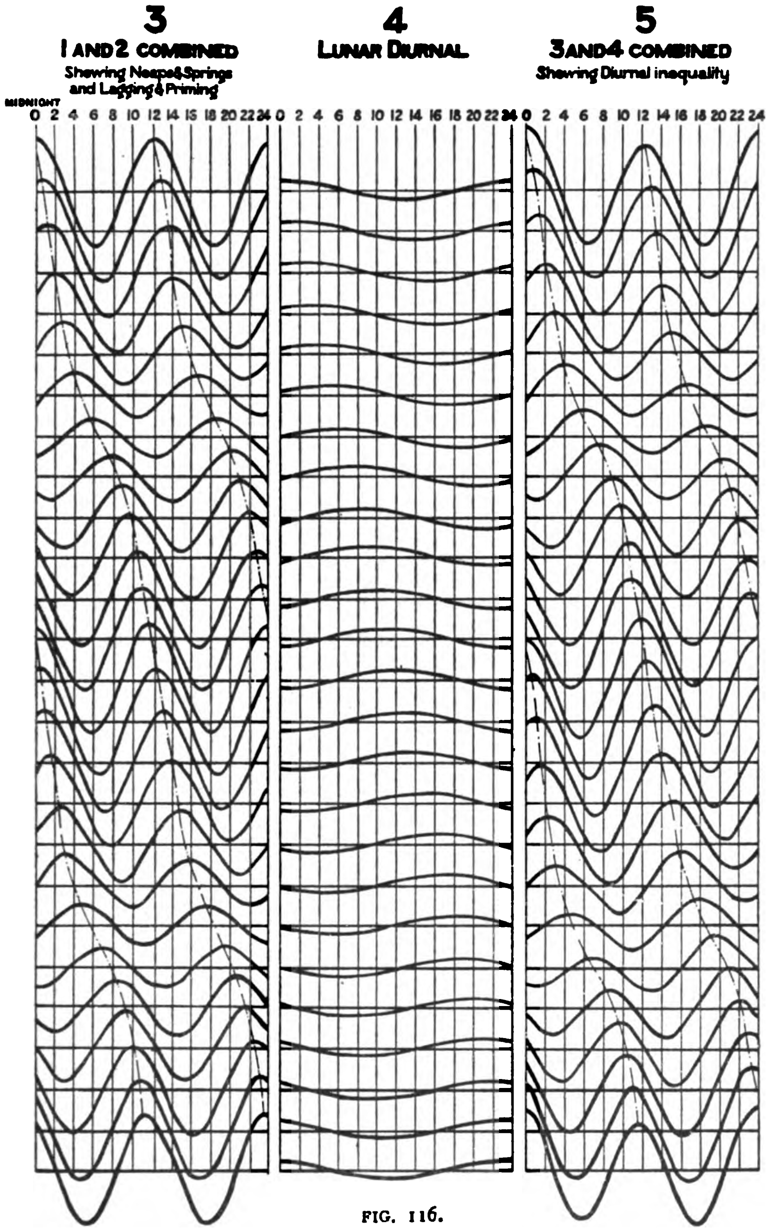


FIG. 116.

Sketch in a fair curve through the points thus determined, a compound harmonic curve will be obtained, and the Datum or zero line may then be drawn at its proper distance below mean sea-level. If the time-intervals be properly

chosen, the resultant curve will represent closely the curve that is produced by plotting actual hourly tidal observations. Any number of harmonic curves may be combined with each other, so that almost any tide-curve may be reproduced, with accuracy. For example, a semi-diurnal harmonic might be combined with a diurnal solar and lunar tide. In this case, the twelve divisions of the generating circle would represent intervals of two hours, and the horizontal time spaces must be twice the length of those used for the semi-diurnal tides.

Example of Combination of Harmonic Curves.

Fig. 115 gives an example of the effect produced by the combination of several simple harmonic curves.

These curves are placed, overlapping, one below the other, so as to give, in moderate space, the effect that may be obtained by combining them. The nature of each series of curves is as follows.

The first series of curves represents the semi-diurnal lunar tides. For convenience in drawing, it has been assumed that the lunar day is exactly 25 solar hours. The error in this assumption, however, involves no error in principle, though incorrect in practice.

It will be observed that the time of high water of the lunar semi-diurnal tide is exactly one hour later, on each successive day.

The second series of curves represents the solar semi-diurnal tides. High water and low water occur at the same instant, day after day, so that the successive instants of these phenomena occur at the same mean times day after day, whereas, in the case of the semi-diurnal lunar tide-curve, the high waters and low waters occur later, day after day.

The third series of curves is a combination of the first and second series, obtained by adding or deducting the ordinates of the two curves to or from each other. The resultant curve indicates the phenomenon of 'neaps' and 'springs,' also that of 'priming' and 'lagging.'

It will be observed that the lines, traced through successive instants of high and low water, are not, as in the case of the semi-diurnal lunar and solar tides, straight lines, inclined in the former and vertical in the latter case.

The fourth series of curves is merely a repetition of the third, for convenience of comparison (fig. 116).

The fifth series of curves is that representing a *diurnal* lunar tide, one having one high and one low water, in the course of twenty-four lunar hours.

It has the same characteristics as the *semi-diurnal* lunar tide, with the exception that there is but one high water and one low water, in the course of twenty-four lunar hours.

The sixth series of curves is the resultant curve, obtained by the combination of the curves one, two and five (or by that of three and five). It exhibits clearly the diurnal inequality, produced by superimposing a diurnal upon two semi-diurnal tides. The resultant curve resembles tide-curves that have been observed in many parts of the world.

The results of observation may be followed and duly represented by the addition and combination of other curves. One could combine a diurnal solar curve, a fortnightly lunar curve and a monthly lunar curve, representing the per-

turbations produced by the difference of the moon's distance from the earth, and so on.

By the assumption that any observed tidal oscillation may be resolved into a number of simple oscillations, having known periods, but epochs and amplitudes determined by observation, the most complex phenomena may be analysed and predicted.

Again, it is perfectly possible to predict the times and heights of high water, even far up estuaries and rivers, under conditions, such that the rise of the tide occupies, perhaps, but three hours out of the twelve. It is only necessary to introduce a sufficient number of tides, or in other words to imagine a sufficient number of attracting bodies. In certain cases it is necessary to suppose the existence of separate tides, having a period of six, four or three hours, solar, sidereal or lunar. Again, long-period tides, of one year, may be required.

If only a sufficient number of harmonic waves are assumed, a resultant curve of any desired complexity, and agreeing with the observed curves, may be obtained.

Analysis of Tidal Observations.

No attempt will be made to describe the method of analysing tidal observations, that is to say, deducing from them the various components. The process is not difficult, but is extremely operose. The number of constants required, usually amounts to 24, and for the Indian and other tables, it is proposed to increase the number to 36. Another reason for omitting a detailed description of the analysis of observations is, that constants would be of little use to the surveyor, because the direct calculation of the predicted times and heights of high and low water is far too lengthy for him to undertake. Such predictions are in practice effected by a machine, an idea of which will be given later on.

The following considerations will give a crude idea of the process of analysis.

The hourly readings of the gauge, either taken directly or scaled from the curve of a self-registering gauge, are tabulated in vertical columns. Each vertical column represents twenty-four mean solar hours, and is numbered from 0 at the top to 23 at the bottom. There are 369 columns, each representing a mean solar day. In each column the tide level is inscribed against its appropriate numbers. The columns are then summed vertically, and also horizontally. The sum of each day's readings appears at the bottom of the columns. The sum of all of the readings for 0 hours, 1 hour, etc. are inscribed at the right of the table. The sum of the sums of the vertical and horizontal summations must agree, if the arithmetic is correct.

The sum of all the sums, divided by the total number of observations, gives, at once, mean sea-level, on the scale of the tide-gauge.

Dividing the sum of each horizontal line, by the number of observations in that line (369) then the average level, at the solar hour represented by that line, is obtained. Thus the mean of all observations, taken at 3 P.M. on consecutive days, gives the mean level of the sea at 3 P.M.

Now the moon and its derivatives have, during the 369 days, been in all possible positions with regard to the sun so that their perturbing influences,

sometimes adding, sometimes deducting, balance each other. The result is that the means of the twenty-four horizontal columns, represent the ordinates of the primitive semi-diurnal solar tide, and at once give one component.

If the means for the hours 0 to 12 differ from those for the hours 12 to 24, this will also give a diurnal tide.

Suppose that, having got a continuous tidal curve, we lay off shorter intervals representing lunar hours, and proceed in the same way.

We can thus obtain the averages for lunar hours, and the elements for semi-diurnal and diurnal lunar tides, and so on for longer periods.

Similar constructions give other components.

The table on opposite page gives the components for several places.

The Tide-Predicting Machine.

Having obtained the 'components,' that is to say, the range of the various tides, generated by the imaginary stars, and their relative positions on a given day, the prediction is effected by means of the tide-predicting machine, the principle of which may thus be briefly described. The tracing of a simple harmonic curve, by means of a crank and slotted link, has been already mentioned.

Cranks and slotted links are provided, equal in number to the number of components to be used. The crank-axles of the several cranks are geared to a common shaft by means of toothed wheels, so proportioned that each crank revolves at a rate, proportioned to the astronomical rate of revolution of the imaginary star which it represents, movement of the common driving-shaft being taken as unity. The lengths of the cranks are adjustable, and they may be clamped in any desired angular position on their crank-shafts.

Each slotted link carries a pulley, attached to the summit of the vertical spindle which determines its up and down motion.

A fine cord, fixed at one end, passes down under the first moving pulley, over a fixed pulley, under a second moving pulley, and so on, to the end of the whole series of moving pulleys, finally passing over a fixed pulley, and carrying at its free end, a weight with a pen or pencil, marking on a drum, geared in proper ratio to the common driving-shaft.

Each moving pulley, as it ascends and descends, owing to the revolution of its crank, describes a simple harmonic motion. The continuous cord combines the motions of all the pulleys, so that the resultant motion of the marking-pen, traces a compound harmonic curve on the paper as it revolves.

The lengths of the several crank-arms are adjusted, by means of a micrometer screw, so that each is proportional, on some common scale, to the range of the component tide. The angular positions of the several crank-arms are then adjusted so that each occupies the angular position of the component or phase of the imaginary stars which it represents, at some given instant of time. The machine is then set in motion. The pencil traces on the drum a curve representing exactly the curve which would be traced by a self-recording gauge, during any period. From this curve the times and heights of high and low water may be scaled off and tabulated. The curve for a whole year is traced in about four hours.

Table of Mean Amplitudes and Epochs of Tidal Constants (for Aden, Hong Kong and Liverpool) forming the Settings of the Tide Predictor.

Period.	Description.	Conventional Letter.	Movement of Imaginary Heavenly Body in Degrees per Mean Solar Hour.	Aden. Amplitude or Semi-range in Feet.	Aden. Epoch or Hour Angle of Maximum Effect of the Component.	Hong Kong. Amplitude.	Hong Kong. Epoch.	Liverpool. Amplitude.	Liverpool. Epoch.
					°		°		°
Sidereal day.	Luni-solar diurnal . .	K_1	15·041	1·30	35	1·19	297	0·36	192
½ (Sidl. day).	Luni-solar semidiurnal .	K_2	30·082	0·20	240	0·16	289	0·94	1
½ (Lunar day).	Lunar semidiurnal . .	M_2	28·984	1·57	226·5	1·43	266	9·98	320·6
¼ (Lunar day).	First overtide of semidiurnal	M_4	57·968	0·01	313	0·08	320	0·69	211
⅙ (Lunar day).	Second " "	M_6	86·952	0·01	345	0·01	113	0·20	331
	Smaller elliptic semidiurnal	L_2	29·528	0·04	225	0·04	264	0·53	329
	Larger " "	N_2	28·440	0·43	221	0·26	255	1·90	299
	Lunar diurnal (declinational)	O_1	13·943	0·66	37	0·86	248	0·37	38
	Solar " "	P_1	14·959	0·39	31	0·38	285	0·13	182
Solar day.	Solar diurnal (chiefly meteorological).	S_1	15·000	0·09	171	0·03	134	..	..
½ (Solar day).	Solar semidiurnal . .	S_2	30·000	0·69	246	0·56	292	3·16	6
	Smaller elliptic lunar diurnal	J_1	15·585	0·09	46	0·04	280	very small	
	Larger " "	Q_1	13·399	0·15	38	0·14	232	very small	
	Larger elliptic solar semidiurnal.	T_2	29·959	0·05	231	0·04	281	0·24	227
	Variational lunar semidiurnal.	U_2	27·968	0·08	190	0·07	239	0·26	34
	Larger evectional lunar semidiurnal.	V_2	28·513	0·10	225	0·11	290	0·53	286
	Compound luni-solar ¼ diurnal.	MS_4	62·032	0·01	155	0·06	358	0·41	258
	Compound luni-solar ½ diurnal.	2 SM	31·016	0·02	101	0·01	3	not evaluated	
	Larger elliptic semidiurnal, second order.	2 N	27·895	0·09	191	0·05	272	not evaluated	
	Compound elliptic lunar (¼ diurnal).	M N	57·424	0·04	17	0·04	180	not evaluated	
	Compound luni-solar declinational (⅓ diurnal).	$M_2 K_1$	44·025	0·02	265	0·04	35	not evaluated	
	Ditto	$2 M_2 K_1$	42·927	0·01	9	0·02	227	not evaluated	
Solar year.	Solar annual (chiefly meteorological).	Sa	·041	0·37	355	0·44	226	0·40	250
½ (Solar year).	Solar semi-annual (chiefly meteorological).	Ssa	·082	0·13	128	0·10	90	0·16	201

The principal components, in general, taken in order of their effects are—lunar and solar semidiurnal, larger elliptic semidiurnal, luni-solar diurnal, and lunar and solar diurnal (declinational); but, as will be evident from the table, this order does not hold universally.

The "epoch" of the component means the angle between the meridian of the place and the imaginary heavenly body which produces the component, at the instant when the latter is at its maximum.

Thus, dividing the epoch by the hourly motion, we get the time in hours, between the time of transit of the imaginary heavenly body over the meridian of the place, and the instant when the tide due to that component is at its maximum at that place.

The period (in mean solar hours) of any one of the tidal components can be found by dividing 360 by the movement of the heavenly body per hour, given in the fourth column.

Horizontal Movement of Tides. Tidal Currents.

Hitherto the vertical movements, or rise and fall of tides have alone been considered. It may not be out of place to make some observations as to the horizontal movements, or 'tidal-currents.' The tide-wave differs little in its mechanical properties, except in magnitude, from an ordinary rolling wave, such as the ripples, produced by throwing a stone into the water.

In a rolling wave, the particles, in each horizontal parallel plane, move with equal velocities in elliptical orbits. Proceeding downwards, the successive

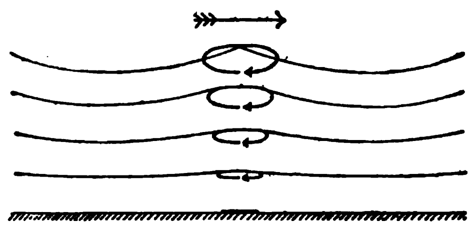


FIG. 117.

particles move in more and more flattened and shortened ellipses, till, at the bottom, the movement is in a horizontal line (fig. 117). The successive particles, in one horizontal plane, taken in the direction of the wave's motion, are, at any common instant of time, successively later in their orbits.

Drawing the section of a rolling wave to a greatly exaggerated vertical scale, the paths of the successive particles may be conveniently shown as circles. Fig. 118 shows the general character of a rolling wave. The successive circles represent the paths of successive particles. Each revolves in its path so that, at its highest point, the particle is moving in the direction of the wave's motion; at its lowest point, in the opposite direction thereunto.

It will be observed from the diagram, that at the crest of the wave, the forward motion of the particles is in the direction of the wave's motion, and is a

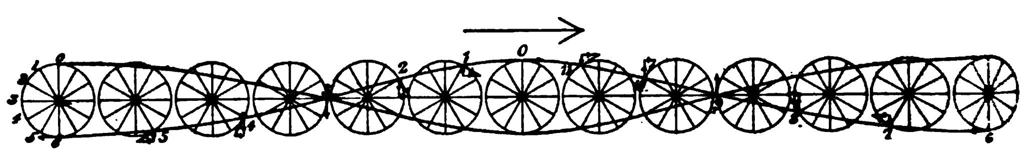


FIG. 118.

maximum in that direction. At its lowest point, the motion is in the direction *opposite* to the wave and at its maximum also. The horizontal motion of the particles, at their mean position, as regards altitude, is zero, but the vertical motion is at a maximum, falling when in front of the crest of the wave, rising when behind it in its line of motion.

Change of Tide Stream in Open Sea.

These considerations lead to the conclusion that in deep and open water, the tide-stream would be in the direction of the advance of the tide wave, and will have a maximum velocity at high water, and would be also a maximum in the opposite direc-

tion at the instant of low water. Further, that there would be 'slack-water,' or no current, at the instants of 'half tides,' that is to say, when the water-level at the point of observation, coincides with mean sea-level. In short, slack-water, in the open sea, occurs at half-tide.

Tidal Motion in Bays and Lagoons.

These simple motions are, however, profoundly complicated by the form of the coast, in bays, estuaries, and rivers. The following considerations, however, serve to give an idea of the phenomena, which may be expected to take place under several conditions.

Let A B (fig. 119) be a straight coast line, plunging abruptly into deep water. Let the direction of the advance of the tide-wave be that of the arrow. At C, assume a small inlet of no great length inland from the coast-line, wide at its mouth, compared with its length inland, like a tidal dock, and being deep throughout in comparison to the rise and fall of the tide outside. Then it is clear that at the inland extremity, there will be no current, the water-level rising and falling to the

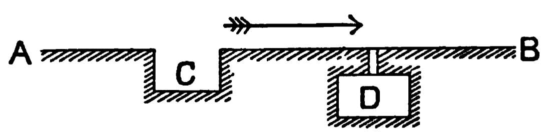


FIG. 119.

same extent, practically as the tide in the open sea. At the mouth there will be a slight in-and-out current, *in* during the 'flood,' or 'rising tide,' *out*, during the 'ebb' or 'falling tide.'

In the open, just outside the mouth of the inlet, the in-and-out going current will be deflected in the direction of the tidal-current outside—the direction of the tide-wave, from half-flood to half-ebb; and against it, from half-ebb to half-flood again.

Next consider a pond or dock connected to the open sea by means of a very narrow channel, such as a pipe or culvert, of so small a section as to require an appreciable head of water to establish a strong current through it, so much so, that during the rise of the tide, the level of the water in the basin never rises up to or falls down to the level of the sea at high- or low-water outside.

It is clear that the instants of high and low water will be later inside the basin than outside. By the time that there is high water outside, the water inside will only have risen to some lower level. The inflow will continue for some time after high-water has occurred outside, and until the level of the water outside has fallen to the level of the still rising water inside, when the current through the channel of communication will stop and reverse its direction. A similar effect will be observed on the falling-tide. This is a pure case of 'in-pouring.' The change of direction of the current occurs at high- and low-water *inside*. An effect of similar character is actually observed in the case of lagoons, connected by narrow outlets to the open sea. In such cases, the phenomena, are complicated by the fact that the water-way and consequent discharge, varies with the height of the tide.

Tide Motion on a Shelving Beach.

The case of a shelving fore-shore may next be considered. Let fig. 120 represent a section of such a fore-shore. The shaded portion has to be filled and emptied during the rise and fall of the tide. This will occasion an indraught, during the rise and an outdraught during the fall of the tide. These motions will, however, be compounded with the motions of the tide-stream in the offing.

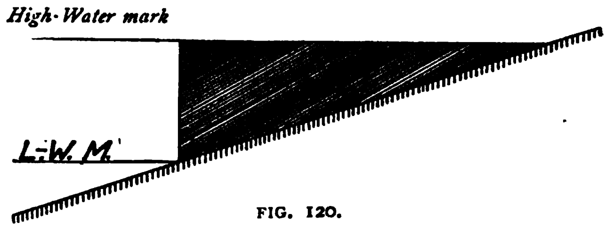


FIG. 120.

The following motions may therefore be expected. Let fig. 121 represent the plan of the coast; the feathered arrow, as before, indicates the direction of the tide-wave in the offing.

At low water, the tidal current will be in the opposite direction to the motion of the tidal wave, and as the water level is, for a short time, stationary, there will

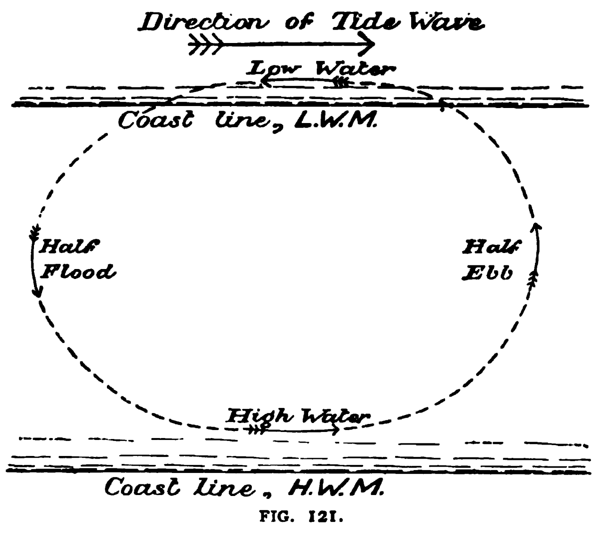


FIG. 121.

be no indraught. At half-flood there is slack-water in the offing, but the tide is still rising, with its maximum vertical motion. There will, therefore, be no current, parallel to the shore, but simply an indraught, perpendicular to the shore-line. As the tide continues to rise, that is after half-flood, the current, parallel to the shore, accelerates. The inpouring retards until, at high water, the current will be parallel to the shore, and in the direction of the advancing tide-wave. As the tide falls, the outpouring motion again sets in, until at half-ebb the current will be simply outward from the shore.

A floating-body, on a uniformly shelving fore-shore, may therefore be expected to describe an elliptical course.

In nature, however, uniformly sloping and straight fore-shores rarely occur.

In practice the only safe plan is to determine the currents, by actual observation. The abstract considerations, put forward in the preceding sections, only serve to indicate what is likely to happen, and the general character of the phenomena to be looked for.

Tide Motion in Estuaries and Rivers.

The tidal-currents, in estuaries and rivers, are even more complicated, than in the case of a uniformly shelving fore-shore. Nevertheless the consideration of a rolling wave, approaching a shelving fore-shore in a direction perpendicular to the shore line, may be considered with advantage.

It will be observed that, when a rolling wave advances into shallow water, its motion of translation becomes slower. The distances from crest to crest, become shorter. The fronts of the waves become steeper, and finally, the wave curls over and breaks on the beach.

It is as though the major axes of the elliptical orbits of the particles, horizontal in the deep-water wave became more and more inclined to the horizon, as the wave advances into shallow water.

In the case of an estuary, the action of the tide-wave, and of the currents which it induces, is a resultant of the several actions which have been described. These are further complicated when a river, of considerable volume, enters the head of the estuary.

Estuary and Fluviatile Section defined.

A river may be divided into an estuary section and a fluviatile section. The estuary section may be defined as the part nearest to the open sea, where—

(*a*) The bottom is below mean sea level, so that there is a considerable sectional area, at low water.

(*b*) The flow of the river is negligible in volume, compared with the volume of tide-water, which enters and leaves at each successive tide.

The fluviatile section is that in which the bottom approaches to or rises above low water mark, and in which the flow of the river bears an appreciable relation to the volume of the entering and outgoing tide water.

Tidal Motion in the Estuary Section.

In the estuary section the inpouring action predominates. The inpouring effect on the rise of the tide produces an inward current. To generate the current, that is to say to overcome the inertia of the particles of water, and further to overcome the frictional resistance of the sides, there must be a surface slope from the mouth inland. As the tide rises the acceleration becomes greater, and a wave-like action is set up which often rolls onward into the fluviatile section. As the estuary becomes narrower, the kinetic energy of the mass of water increases. The wave action becomes more and more prevalent till at a certain distance up the estuary the level of high water may be actually higher than that of the corresponding tide at the mouth of the estuary. A similar action takes place on the falling tide (or ebb). The results of these actions are—

(*a*) That the instants of high and low water are respectively later, as the estuary is ascended than the corresponding times outside.

(*b*) That in a free and uniformly contracting estuary, the tide-range, at some point near its head, may be actually greater than it is, in the sea at the mouth, high-water mark being higher and low-water mark lower at the upper station than at the lower for any corresponding tide.

(*c*) Proceeding up the estuary, beyond some critical point of maximum range, frictional resistance diminishes the range of the tide, the kinetic energy of the water particles expending itself in friction and eddy-motion.

Tide at Sheerness.

For example, the range of spring tides, at London Bridge is about 22 feet, whilst at Sheerness it is only 15 feet, and at Ramsgate somewhat less again.

(*d*) In the estuary, owing to the predominance of the inpouring action, the direction of the current changes, at or near the instant of high or low water, respectively, at any given point of observation. Nevertheless, it is often observed that the flood stream continues in an up-stream direction, for some time after the instant of high water. In the Thames, watermen often say that "the tide has made its mark on the shore, but is still rising in mid-stream." The fact being that the flood or up-stream current, as indicated by the swinging of vessels at anchor, and the motion of floating objects, continues owing to momentum, for an appreciable period of time, after the water-level has fallen, as indicated by the line of detritus, which is left by the falling tide, on the beach, or "hard." This phenomenon is less noticeable on the ebb.

Tidal Motion in the Fluviatile Section.

The wave, generated by the inpouring action, in the estuary section, rolls on into the fluviatile section diminishing in range, partly on account of fluid friction, and partly from rise in the bed of the river. The distance to which it will penetrate into the river depends upon the area of cross-section and upon the freedom from frictional resistance, owing to bends and other obstructions. In the river the form of the tide-wave alters. It generally becomes steeper in front as it advances. In addition to the principal tide-wave, secondary waves of shorter periods (six, four, and even three hours) are generated, especially in shallow water. The period of rise becomes shorter than the fall. The whole period of rise may occupy three hours or less, the fall nine hours or more. Towards the latter part of the tidal period, the level is nearly constant, coinciding approximately with the flow of the river, or what it would have been had it been free from tidal influence. It seems as though a wave, of short period, had been generated and rolled into the river from the estuary below.

Tide-waves will penetrate up large rivers, and may be perceptible even where the bed of the river is above sea level.

Mean Sea-level. Tidal Observations to be referred to a Datum Plane, such as Mean Sea-level.

The main object of the preceding discussion is to emphasise the necessity for referring the levels of the bed and banks of estuaries and rivers, to some one fixed datum plane. "High-water mark" and low-water mark are not fixed planes, even for one place. The range of successive tides, and consequently the levels of successive high and low-water marks, varies from

day to day, as has been already shown. Mean sea level, on the other hand in the open sea, as the average of continued hourly observations extending over 369 days, is a fixed level all over the earth, it is neither more nor less than the geodetic surface of the earth, as defined in the chapter on this subject (Chapter VII.).

The Disadvantages of referring to Low-water Mark as a Datum.

If tide-gauges be observed at sundry stations, up an estuary, it may often be found that, up to a certain point, the *mean* water level would coincide with mean sea level, at the open seas, though high water might be higher and low water lower, than at the mouth. The limit of coincidence, of mean water level with mean sea level, as determined by accurate levelling, might be used as a definition of the limit of the estuary-section. Beyond this point, the fluviatile section commences. Here mean river level no longer coincides with mean sea level. If the level of the bed, were expressed as depths, below a low-water mark, as is done, for obvious and sufficient reasons in navigation charts, then an erroneous idea of the levels would be formed. The only safe plan, for engineering purposes, is to refer *levels* of the river beds to some fixed plane, such as mean sea level, or to a plane at a fixed distance, above or below it. This done, if a navigation chart is to be prepared a more or less arbitrary plane of low-water may be established for successive reaches of the river, and levels reduced to 'soundings,' or depths below these planes, for purposes of navigation.

If, however, the depths were referred to low or high-water mark, without recording actual levels, erroneous conclusions, as to the effect of engineering works might result. For example, take the case of a small and tortuous stream, choked perhaps with weeds. In the natural state of the rivers the tide would hardly be perceptible in the upper regions, the energy of both the tide-stream and the tide-wave being absorbed by various resistances. Now let the stream be cleared, its course shortened and improved by cutting off bends. The effect of these improvements would be to admit the tide-wave more freely than before. In the upper reaches, the tide would rise higher and ebb lower than before. Vessels of larger draught could ascend the river at high water. If, however, the only record were depths at low water, one might infer that as the low-water depth in the upper reaches had become lessened, the bottom was rising by accretion, whereas, perhaps, the opposite effect was taking place.

Effect of Wind on the Tides.

The effect of winds upon the rise and fall of the tide is a subject on which there is great difference of opinion. The usual belief is that wind has an important influence on tides. Personally, the writer holds the opinion that wind has less influence than is generally supposed. He bases his opinion, firstly, on observed facts. The tides at several ports (Aden, Bombay, Singapore and Hong Kong) where he and others have made observations, present apparent anomalies, one of which is a large diurnal inequality (much larger than is usual in English waters).

The observations do not agree with the rules for tidal prediction usually given in works on navigation. These perturbations were tacitly ascribed by mariners to be due to wind. In several places the variations were ascribed to the change

of the monsoon. There were tides for the 'north-east' and 'south-west' monsoons respectively. Proper continuous observations having been made, however, and subjected to harmonic analysis, coefficients or factors were determined, by which the time and height of high water are now predicted with accuracy, without reference to the effect of winds, though these varied greatly in force and directions at different seasons, and from day to day. Mr. E. Roberts has informed the writer that he has often found that apparently abnormal tides, popularly ascribed to wind, were predictable, when a suitable number of coefficients were used in the proper manner.

The writer cannot conceive that the effect of wind on the surface of water can produce currents sufficiently strong to cause material elevation or depression.

The equinoxes are the times at which theoretical consideration would lead to the anticipation of tides of unusual range. Observation confirms this anticipation. It is a matter of common knowledge that violent winds often prevail at the equinoxes (equinoctial gales). It seems, therefore, that the configurations of the moon and sun, which produce equinoctial spring-tides, also produce atmospheric disturbances. The extraordinary high tides at such times therefore may be traced direct to the heavenly bodies, and not to wind. The writer is fully aware of the fact, however, that extraordinary and unpredictable tidal perturbations, accompany violent winds.

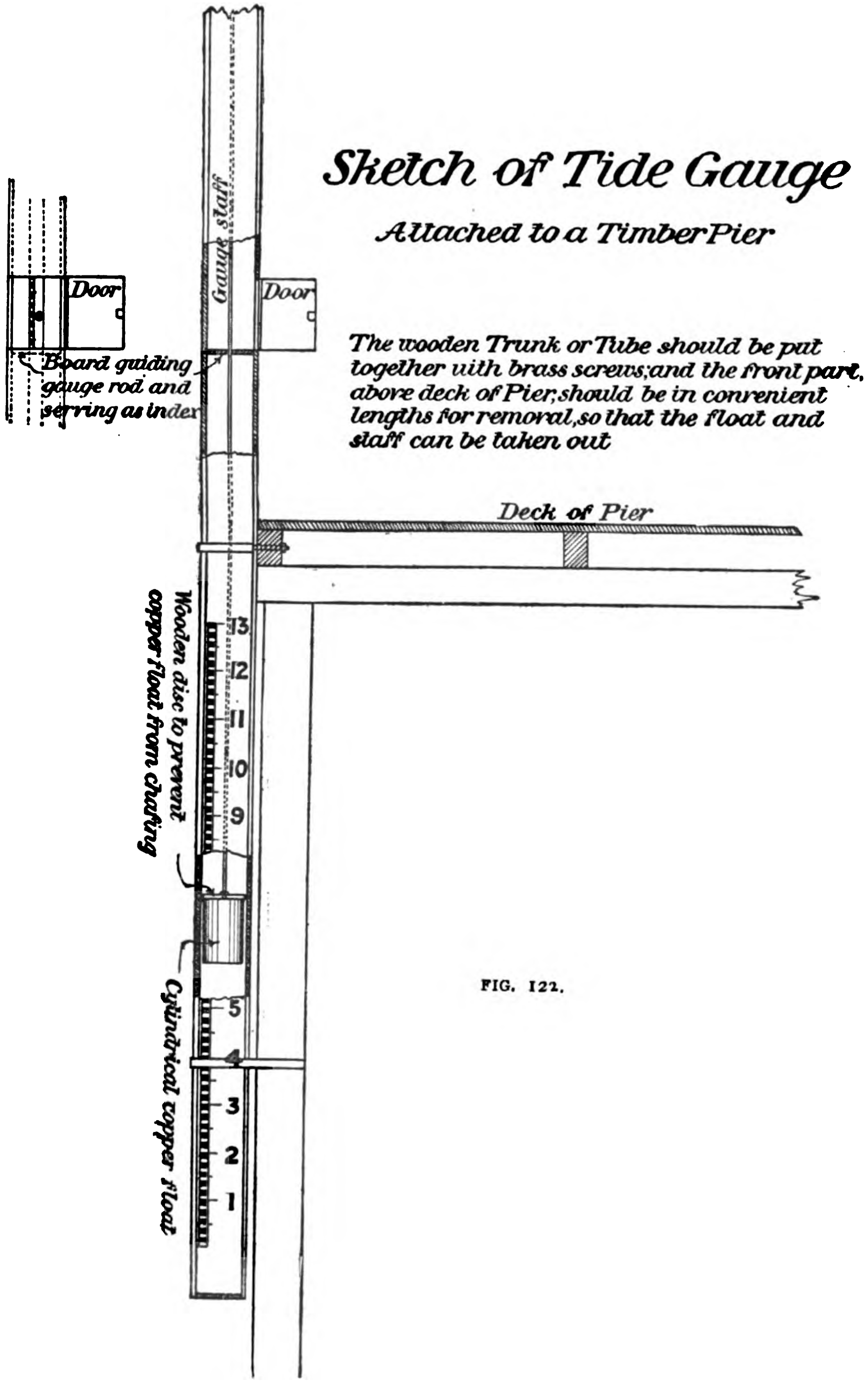
Barometric Effects.

The writer believes that the elevation of the water which in certain cases doubtless takes place, is due to the great depression of the barometer, which actually occurs in the centre of a cyclone as it passes onward. The depression, in one instance within the writer's knowledge, amounted to 2 inches of mercury. That is to say the barometer was 2 inches lower in the centre of the cyclone, than at its margin. Now 2 inches of mercury is roughly equivalent to 2 feet of water; so that, at the centre of the cyclone, supposing it stationary, the sea level would be about 2 feet higher than at its circumference. The reduced pressure at the centre of the cyclone as it sweeps along, may reasonably be expected to produce a large wave-like elevation, following the course of the cyclone. If the great ocean tide wave is only 4 feet, and yet suffices to generate tides, ranging 20 feet and upwards, it is clear that a relatively small barometric wave may produce great variations in the ranges of tides in estuaries.

To conclude, the writer ventures to believe that wind in itself has little effect on the tide, but that large barometric depressions which usually accompany violent winds, may on the other hand have an important effect, though such effects are relatively of rare occurrence.

On Tide-Gauges. The Float and Pole Gauge.

Fig. 122 shows a simple form of tide-gauge which has been found useful. It is suited for attachment to a pier or jetty extending into deep water, at low-water spring tides. The rod carried by the float, is graduated from above downwards, and read by means of a fixed index, rendered visible by an opening or door in the side of the case. The case contains also a clock, and a lamp for reading at night. To obtain records with such a gauge, a man must be in attendance night and day.



Sketch of Tide Gauge
Attached to a Timber Pier
The wooden Trunk or Tube should be put together with brass screws; and the front part, above deck of Pier; should be in convenient lengths for removal, so that the float and staff can be taken out
Gauge staff
Door
Door
Board guiding gauge rod and serving as index
Deck of Pier
Wooden disc to prevent copper float from chafing
Cylindrical copper float
13
12
11
10
9
5
4
3
2
1

FIG. 122.

The zero of the outside scales should be referred by levelling to one or more permanent bench marks on shore. Then by reading the outer scale in calm weather, the index of the float staff can be set so that the readings coincide.

The float staff is graduated from above downwards. This is, on the whole, the most convenient plan, because the reading always takes place at the same level. It has the disadvantage that inversions in reading the smaller subdivisions may take place, for example 9·3 might be recorded as 8·7. This has not in the writer's experience proved to be an important defect, for these errors may easily be detected and corrected. It might, however, be well to number each tenth as though it were a unit. For 9·3, 93 would be inscribed.

The Self-registering Tide-gauge. The self-registering tide-gauge is exceedingly simple in principle, though its use is not so elementary in practice.

A float, of large diameter in comparison with its depth, swims in a cylinder or well, having a free communication with the sea, placed well below the lowest low-water mark. The opening must have sufficient area to insure that the water level in the well is, at all states of tides, identical with the sea level outside. At the same time, the opening should be small enough to prevent ordinary wind-waves from creating oscillations of short periods in the float-well. A cord, attached to the float, passes over, or is coiled round a wheel, being maintained, in constant tension, by means of a counter-weight. On the spindle of the wheel a smaller wheel is keyed. Round the latter a cord winds which moves a sliding pencil-carrier. This cord is kept in tension by a counter-weight.

The pencil traces a curve, upon a sheet of paper, clamped on to a horizontal drum, which may be 2 to 3 feet long, and which is caused to revolve about its axis, *once in twenty-four solar hours by means of a good clock.* The pencil thus traces, to a reduced scale, the tide-curve. Usually this paper remains on for a week or a fortnight, and the result is a series of interlacing curves.

An improved tide-gauge, designed by Lord Kelvin, is described in the 'Proceedings' of the Institution of Civil Engineers, Vol. LXV. Part III.

Thus far all seems simple with respect to general mechanical arrangements. The writer has, however, found by experience, that the self-registering tide-gauge, even in its most improved form, is by no means an instrument that can be left to itself. The cord may slip, stretch, or break, the pencil-carrier may jam, and the clock may stop. The presence of an observer is therefore necessary, who must, from time to time, correct the clock, and he must be diligent in comparing the indications of the pencil on the paper with some simple positive gauge such as a graduated scale outside the float well. Efficient self-registering tide-gauges are, moreover, costly, and their proper installation involves substantial and expensive works. For ordinary purposes, where skilled supervision is scarcely obtainable, and repairs to clockwork can be effected with difficulty, the writer believes that the float and staff will, in most cases, give practical accuracy at the minimum of cost, worry and annoyance to the surveyor.

Though a year's observations will determine the constants fairly accurately, still it is satisfactory to have a second year of records as a check, and for the readjustment of constants. Three years' observation at the outside will give all

the information for the practical prediction of the tide and for the determination of mean sea-level. After this time, further observations are merely of academic interest, such as the determination of tides of very long periods, the effect of great atmospheric disturbances, or the secular rise or fall of the coast. It does not seem therefore desirable to procure a costly instrument when it only need be used for a relatively short time.

Tide-gauges on Open Coasts.

It may be desired to take tide observations on an open beach, shelving gently seaward. To construct a jetty with a substantial head planted below low-water spring tides, for the reception of the tide-gauge would probably be impracticable on account of cost, or for the same reason to construct a low-level channel from a well on shore would also be

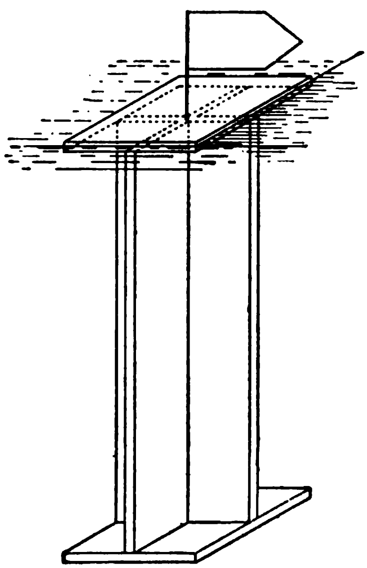


FIG. 123.

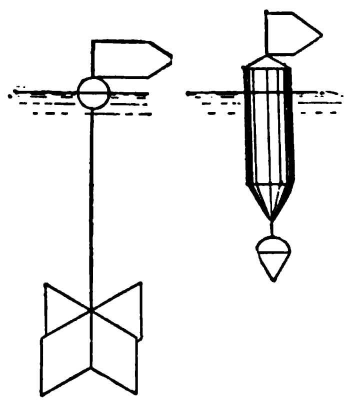


FIG. 124.

impracticable. In such cases a syphon may be employed. The summit of the syphon pipe should be a foot or so below high-water mark neap tides, and a tap should be provided at the highest point to discharge air, and some arrangement should be made for scouring out silt which might accumulate in the pipe. (*Vide* 'A Manual of Tidal Observations,' by Major Baird, R.E., London, Taylor and Francis, 1886.)

A rod and float tide-gauge, similar to that already described, should be attached to a substantial pile, driven about low-water neaps, so that a check reading, can be obtained at any time, except that of extreme low-waters. If the levels as recorded in the open water do not coincide with the levels in the float cylinder, it is a sign that the pipe requires cleaning, or that air has accumulated in the summit of the syphon.

Determination of Direction and Velocity of Tidal Currents.

It is often necessary to determine the direction and velocity of tidal-currents, such a case often occurs in connection with projects of sewerage. It is necessary to trace the path of the sewage at different states of tide when discharged from a proposed point of outfall. For this purpose, a number of floats must be prepared. Fig. 123 shows a form which has been found convenient. These floats are put in at the point to be investigated in succession at intervals of about an hour, and allowed to float away freely. They are then followed by a steam launch and their positions determined by sextant angles to known objects on shore, or these

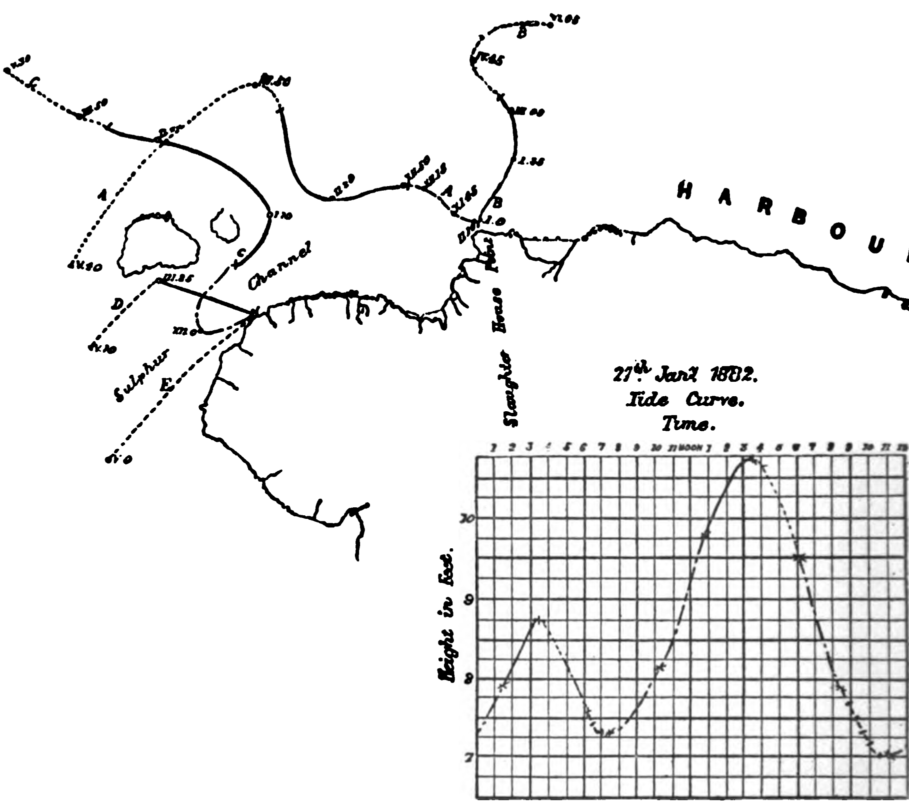


FIG. 125.

positions may be taken up by observations from two theodolites, or two plane-tables on land. Each position is plotted forthwith, and the time of determination is noted against the point on the plan representing the corresponding position of the float. Through the points representing the successive positions of each individual float, a fair curve is drawn to represent its path.

This is done with each of the floats.

Meanwhile, a tide-gauge is read continuously at some base-station, and the hourly readings plotted as a curve. The path of each float is then determined,

and graduated into hourly spaces. The points so obtained represent the position of the float at equal intervals of one hour, and show the direction of its motion at the corresponding hourly heights of the tide. For recording the direction of the deep water flow the float shown in fig. 124 is useful.

Fig. 125 shows an example of this procedure. It is a copy of one day's observation taken at Hong Kong for the purpose of determining the suitability of a certain point of sewage outfall. In the corner of the drawing, the tide-curve for the corresponding day is shown.

These observations are to be repeated frequently at different periods of the lunation, both at neaps and springs, so that the currents were determined at all states of the tide.

Combining a number of such observations, a general diagram could be prepared, showing the direction of the currents on the flood and ebb respectively.

APPENDICES.

APPENDIX A.

NOTATION ADOPTED.

A. Azimuth.

Æ or R.A. Right Ascension.

Ast. T. Astronomical Time.

E. Equation of Time.

L. Longitude (E. or W.).

ϕ Latitude (N. or S.).

P.V. Prime Vertical.

R.O. Referring Object.

T. Watch Time of Transit.

T_0 Computed ditto.

L.S.T. Local Sid. Time.

G.S.T. Greenwich ditto.

L.A.T. Local Apparent Time.

G.A.T. Greenwich ditto.

L.M.T. Local Mean Time.

G.M.T. Greenwich ditto.

L.S.N. Local Sid. Noon.

G.S.N. Greenwich ditto.

S.T. Siderial Time.

L.A.N. Local Apparent Noon.

G.A.N. Greenwich ditto.

L.M.N. Local Mean Noon.

G.M.N. Greenwich ditto.

h Altitude.

h^1 Apparent Altitude.

$m = \dfrac{2 \sin^2 \frac{t}{2}}{\sin 1''}$. (*Vide* Table V.)

m_0 = Mean of Values for m.

p or Δ Polar Distance.

t or H.A. Hour Angle.

α Right Ascension (*sometimes used*).

γ First Point of Aries.

δ Declination.

ζ or z Zenith Distance.

ζ_0 or z_0 Mean ditto.

ζ_1 or z_1 Approximate ditto.

APPENDIX B.

Form for Entries in Angle Book (Field Note Book).

Sextant (or Theodolite). — Observations for Latitude (Time or Azimuth) to Sun (or Star and name). — Page.........

Place.......................... Bar.............. — S. or M.T. Watch or Chronometer No.......... — Rate {Gaining......... / Losing.......... — R.O........................

Date.......................... Therm. .... — Mag. bearing of R.O....

Level Value (for Theodolite)......... (per division) — † R.A. of Star (S.T. of Transit).................. — Error {Fast............ / Slow............

Index Correction (for Sextant)....... — † Watch or Chron. Time of Transit............. — T_0 ....

Limb (apparent as in field of views) of Sun or Star observed.	* Face.	Vertical Angles.								Level.		Watch or Chron. Times.			Horizontal Angles.													
		A. Vernier. B.					Mean of A and B.			E	O				A. Vernier. B. † (or $T_0 \sim T_0$). † (or m).					Means of A and B.			Angle Diffce. R.O. & Star.			Means of Pairs to R.O.		
		°	′	″	′	″	°	′	″			h.	m.	s.	°	′	″		″	°	′	″	°	′	″	°	′	″

Totals . . .

(State if eye-piece reversed or not.) Approximate Means .

Index Correction or Local Correction } ±

Corrected Double (or Single) Altitude .

+ Sum

+ Mean

Angle between R.O. and Star {Sum / Mean

* Face = Vertical Circle R. or L.

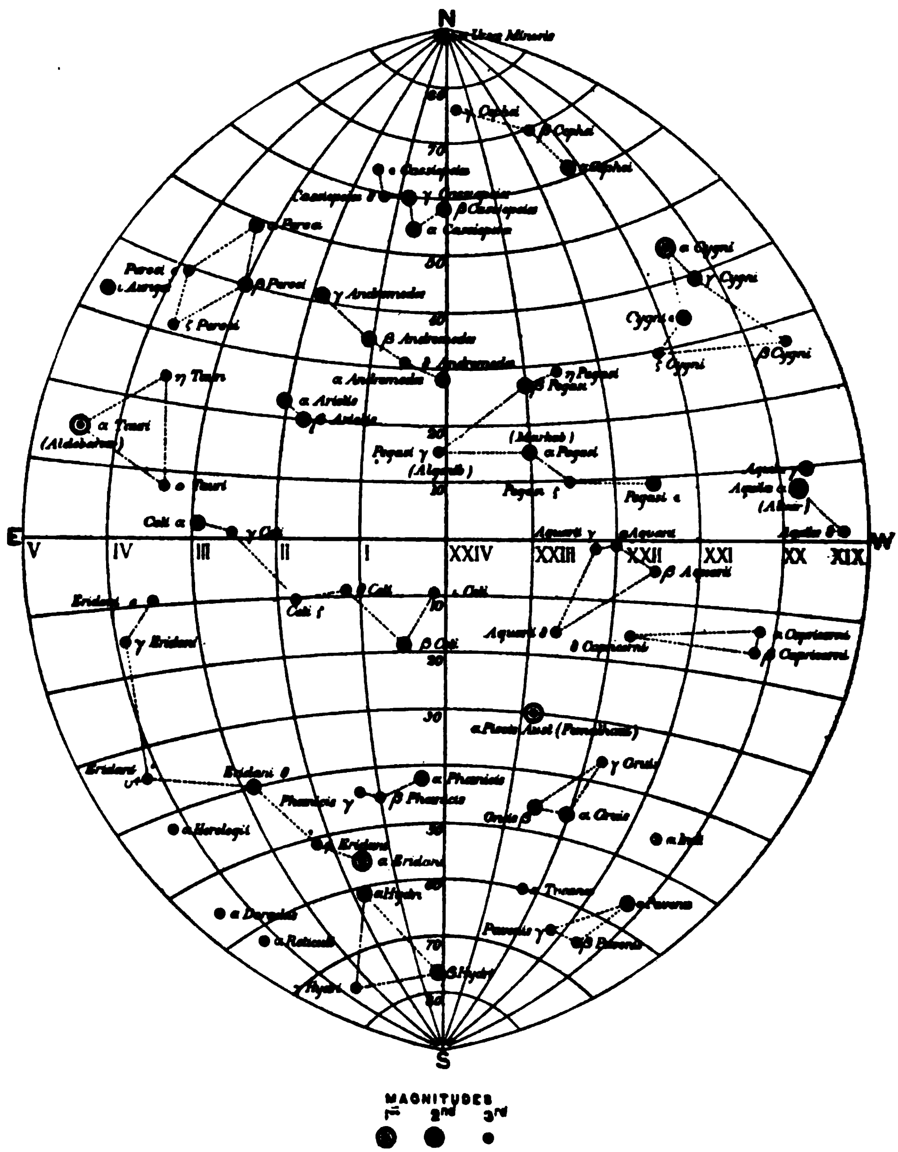
† Only required for Circum-Meridians.

Observer...............................

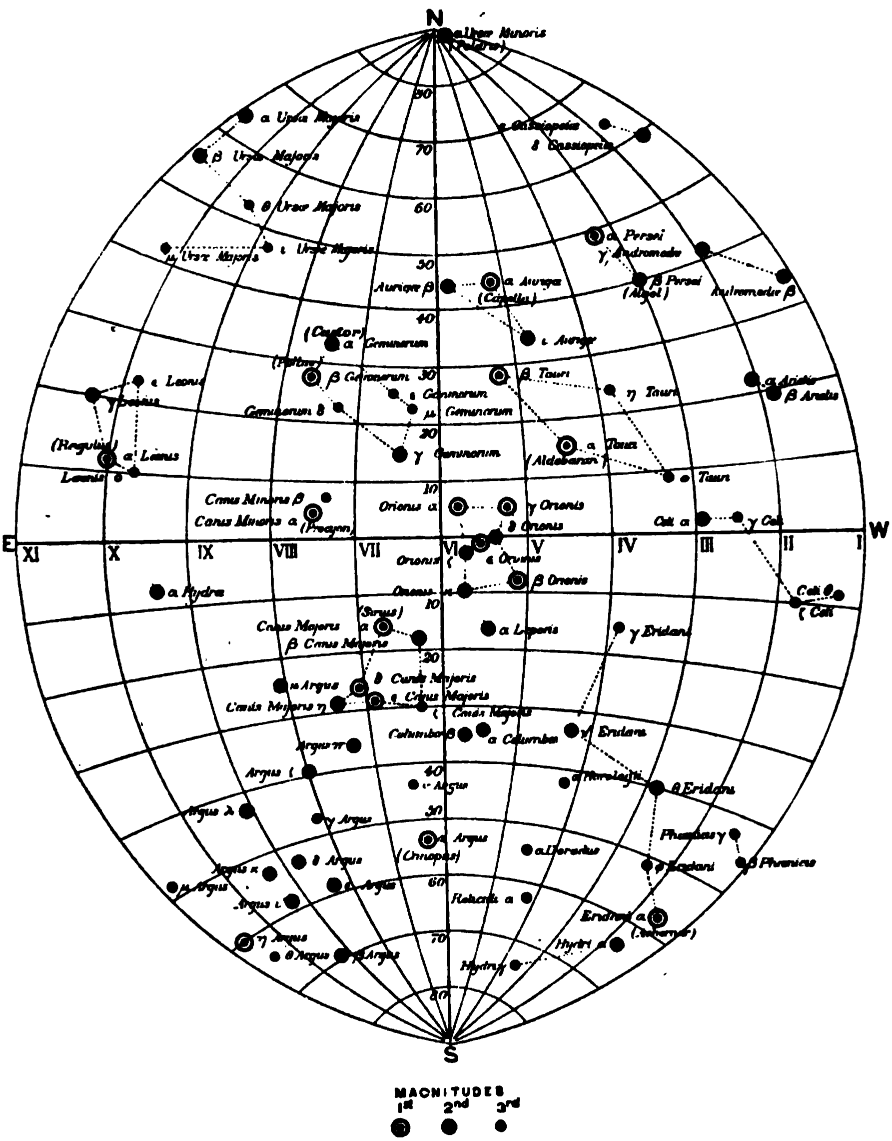
Booker.. ....

APPENDIX C.

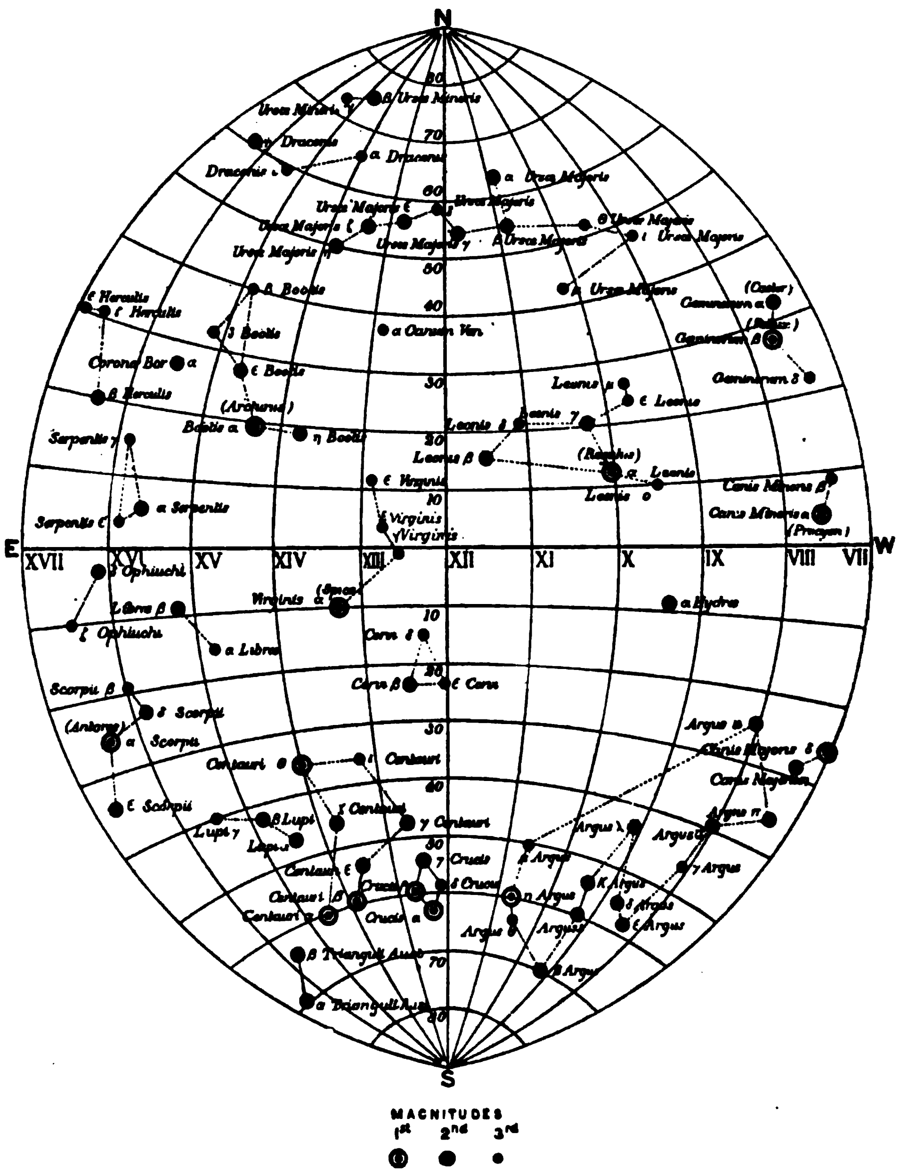
Star Chart I.



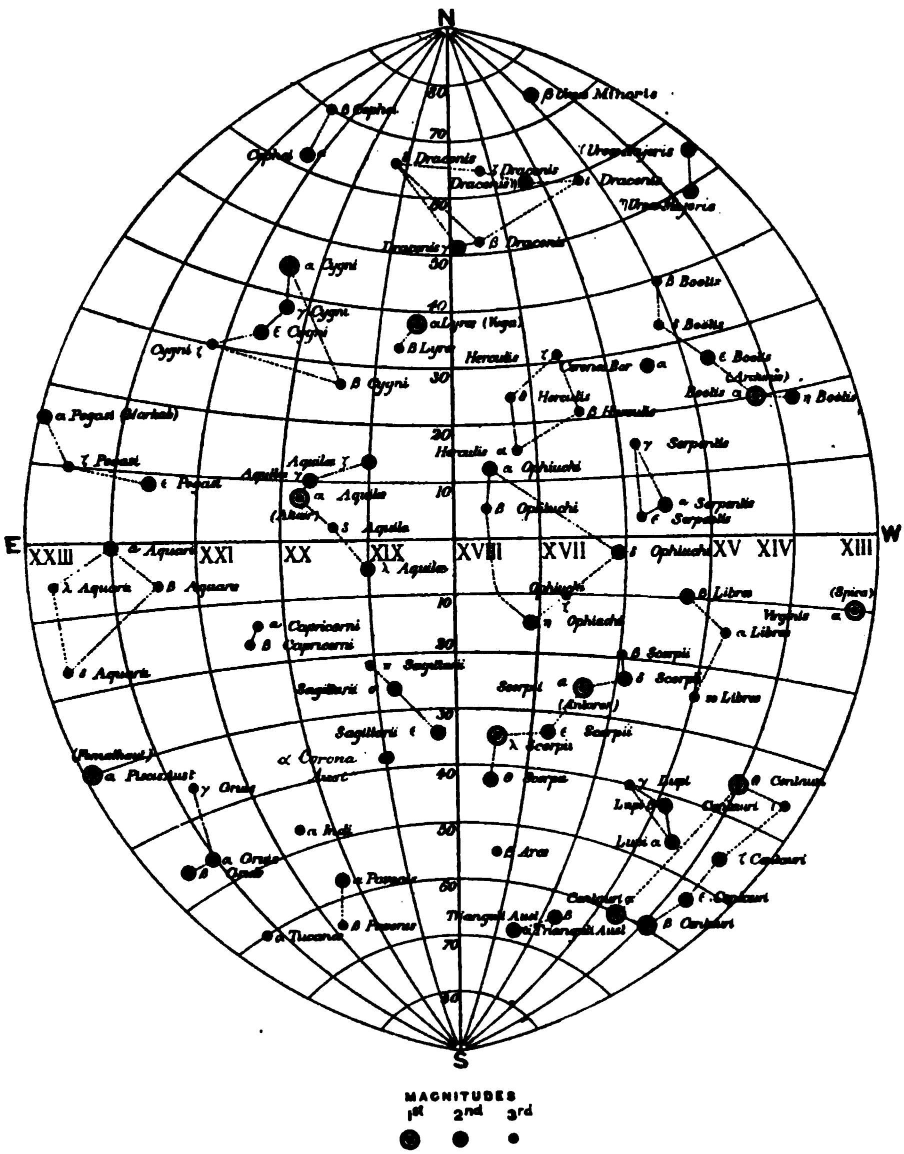
STAR CHART II.



STAR CHART III.



STAR CHART IV.



APPENDIX D.

Form I.—Latitude by Circum-Meridian Altitudes of the Sun or a Star.

Star (or Sun) . ...

Place ... .

Date.............................

Approx. Lat.................

Approx. Long..................

Chronometer Sid. T. (or M.T.)............

Correction {Fast − .... / Slow + ...}

Rate {Gaining.. ...... / Losing ....}

Sextant No............... or Theodolite No...........

Bar. ....

Therm. ....

Observed Double (or Single) Alts. of Star (or Sun's Limb).	Chronometer Times.	t Hour Angles $T_0 \sim T$.	m or $\frac{2 \sin^2 \frac{t}{2}}{\sin 1''}$	To find G.M.T. of Mean of Observations.		To find z_0 Mean Z.D.	Calculations to find $A\,m_0$.		Final Correction.	
1				L.A.T. in M.T. Intls.		Apply Corrs. to Means marked *	log cos ϕ_1 . .		z_0 =	
2				Equation of Time	±		log cos δ . .		$A\,m_0 = (-)$	
3				L.M.T.		° ′ ″	log cosec z . .		Z.D. (meridian)	
4				Corr. for Long. . .		=	log A =		Dec.	
5				G.M.T.			log m_0 =		True Latitude ϕ	
6										
7				Greatest Alt. corr. for Double Refn., Semi-diam., and Parallax	° ′ ″ 90 0 0		log $A\,m_0$ =			
8							($A\,m_0$ is + Sub. Polo)			
9				Zenith Dist. z . .						
10				Declination δ . .						
11				Approx. Latitude ϕ_1 .						
12										
Sums	red. to M.T. interval if necessary. Sums									
* Means	Means		m_0							

N.B.—Lat. and Dec. are + N and − S of Equator. Zenith Dist. is + with Star S and − Star N of Zenith.

$z_0 = \zeta_0 - A\,m_0$ and $A = \frac{\cos\phi \cos\delta}{\sin \zeta_1}$.

With Star compute Chron. Time of Transit.

If with M.T. Chron.

	h. m. s.
R.A. of Star, i.e. S.T. of Transit . . L.S.T. of L.M.N.	
G.S.T. of G.M.N.	
Corr. for Long. E. or W. (− or +). at 9·86 seconds per hour . . .	
∴ Sid. Int. of Transit from L.M.N.	
M.T. Equivalents {h. m. s.	
∴ L.M.T. of Transit	
Chron. Error	
∴ Chron. Time of Transit T_0	

With Sun compute Chron. Time of Transit.

If with Sid. Chron.

	h. m. s.
Sun's R.A. (Page I of N.A.)	
Correction for Long. E. or W. (Long. in hours × hourly var. of R.A.). Long. E (−). Long. W. (+)	
Chronometer Correction to S.T. . . . . . ±	
S.T. Chronometer Time of Transit	

If with Mean Time Chron. .

	h. m. s.
L.A.T. of L.A.N.	
Equation of Time	
Correction to ditto	
L.M.T. of L.A.N.	
M.T. Chronometer Correction	
M.T. Chronometer Time of Transit.	

Form II.—Latitude by Altitudes of the Pole Star (Polaris).

Place.................. Approx. Long. E. or W................ Bar..............

Date.................. Therm.........

Chron. (S. or M.T.) {No................ Corr............ Rate..............

Elements.

	h. m. s.
With M.T. Chron.	
G.S.T. of G.M.N. . . .	
Corr. for Long. {E − W + . .	
* L.S.T. of L.M.N.	
Mean of Observed Times . .	
Chron. Corr. or L.M.T. {Fast − Slow +	
L.M.T. of Obsn.	
S.T. Equivalents {hrs. . . mins. . . secs. . .	
Sid. Interval from L.M.N. .	
L.S.T. of L.M.N.	
* L.S.T. of Obsn.	

Refraction.

	° ′ ″
Refraction due to Observed Mean Altitudes	
Corr. for Bar.	
Corr. for Therm.	
Corrected Refraction . . .	

	° ′ ″
Obsd. Mean Altitude	
Corrected Refraction . . .	
* True Mean Altitude	

Computation by Tables in the N.A.

	° ′ ″
* True Mean Altitude	
Subtract 1′	1 0
Reduced Mean Altitude . .	
1st Corr. with argt. L.S.T. . . ±	
Approxte. Latitude . . .	
2nd Corr. with argt. L.S.T. and True Alt. . . . . +	
3rd Corr. with argt. L.S.T. and date . . . . +	
True Lat. N. or S. . . . ±	

Elements.

	° ′ ″
Dec. (δ)	
Correction for Long. E. or W. (L in hrs. × hourly var. . . .)	
Corrected δ	
	90 0 0
Polar distance (p)	
p in seconds . . .	00

	h. m. s.
* L.S.T. of Obsn.	
R.A. of Star	
Hour Angle (t) in S.T. . . .	
if t is (−) subtract from 24 hrs. .	
	15
t in arc	

Logarithmic Computation.

log p (in secs.)	
log cos t	
if t is in 2nd or 3rd quadt. cos t is (−)	
log 1st Corr.	
do. in secs. . . . . ±	
log p (in secs.)	
log sin t	
log p sin t . . . . =	
	2
log p^2 sin^2t	
log tan h	
log sin 1″ ($\bar{6}$·6855749)	$\bar{4}$·6855749
Av. Comp. of log 2	$\bar{1}$·6989700
log 2nd Corr.	
	15
2nd Corr. in secs. . . . . +	

	° ′ ″
True Mean Altitude	
1st Corr. (appte. sign to cos t). . ±	
2nd Corr. (always added) . . . +	
True Latitude N. or S. . . . ±	

FORM III.—TIME BY ALTITUDES WITH AN E. AND W. STAR.

Place.................... Lat............... Bar............ M.T. / S.T. Chron. No................ Sextant No......... I. Error ....

Date...................... Long. E. or W. Therm........ Rate.............. or Level Divn. Value ....

App. Corr....... Theodolite No.....

Name of E. Star.	Obsd. Alts.	Obsd. Times.	L.S.T. of L.M.N.	Refraction.	Polar Dist.
	° ′ ″	h. m. s.	G.S.T. of G.M.N.	Refraction for Altitude .	Dec. N. or S. ° ′ ″
			Corr. for Long. −E. or +W. at 9·86 secs.	Corr. Bar. . .	90 0 0
			†L.S.T. of L.M.N.	Corr. Therm. .	Polar Dist. . (p)
				Corr. Refn. .	$p = 90 \pm \delta$ If ϕ and δ of same name, then subtract.
Sums					
Means					

Mean Obsd. Alt. .
(Corr. for I. Corr. and ÷ 2 if Sextant)
Corr. for Refrac. .
True Alt (h) . .
Lat. (ϕ). . .
Polar Dist. (p). .
2)

(s) . . . . log cos
($s-h$) . . . . log sin
($s-\phi$) . . . . log cosec
($s-p$) . . . . log sec
2)

long tan $\frac{t}{2}$ =

in arc $\frac{t}{2}$ = 5)
3)

(above) *t in Time (West) s

	h. m. s.
*t in time West	
R.A. of Star	
L.S.T. of Obsn.	
If M.T. Chron. is used. †L.S.T. of L.M.N. . .	
Sid. Intl. from L.M.T. .	
M.T. Equivts. hrs. mins. secs.	
L.M.T. of Obsn.	
Chron. Time of Obsn.	
Chron. Fast or Slow	

Mean Correction of Chronometer.

Name of W. Star.	Obsd. Alts.	Obsd. Times.	Refraction.	Polar Dist.
	° ′ ″	h. m. s.	Refraction for Altitude .	Dec. N. or S. . . ° ′ ″
			Corr. Bar. .	90 0 0
			Corr. Therm. .	Polar Dist. . . (p)
				$p = 90 \pm \delta$ If ϕ and δ of same name, then subtract.
			Corr. Refn. .	*N.B.*—Polar Dist. is reckoned from the Elevated Pole, and if the Lat. and Dec. are of the same name *subtract from*, and if of opposite names *add to*, 90°.
Sums				
Means				

Mean of Obsd. Alt.
(Corr. for I. Corr. and ÷ 2 if Sextant)
Corr. for Refrac. .
True Alt (h) . .
Lat. (ϕ). . .
Polar Dist. (p) .
2)

(s =) . . log cos
($s-h$ =) . . log sin
($s-\phi$ =) . . log cosec
($s-p$ =) . . log sec
2)

log tan $\frac{t}{2}$ =

in arc $\frac{t}{2}$ = 5)
3)

(above) *t in time (West) s

	h. m. s.
*t in Time West	
R.A. of Star	
L.S.T. of Obsn.	
If M.T. Chron. is used. †L.S.T. of Obsn. . .	
Sid. Intl. from L.M.T. .	
M.T. Equivts. hrs. mins. secs.	
L.M.T. of Obsn.	
Chron. Time of Obsn. . . .	
Chron. Fast or Slow . . .	

FORM IV.—TIME BY SINGLE ALTITUDES OF SUN WITH A SEXTANT.

Place............... Lat. .... L.M.T. / L.S.T. } Chron. No. .... Bar................

Date................ Long. (approx. E. or W.)

Sextant or Theodolite. Approx. Corr. ± Index Corr. (if Sextant) ± Therm. ....

Apparent Limb Obsd. to ☉ or ☉	Observed 2 Alts.	Observed Times.	Elements.	
	° ′ ″	h. m. s.	L.S.T. of L.M.N.	Eqn. of Time (E).
			G.S.T. of G.M.N.	E at G.M.N. . .
			Corr. for Long. (− E. + W. at 9·86 secs. per hr.)	Corr. for interval in hours . . .
Sums			L.S.T. of L.M.N.	E at Mean of Observations .
Means	—	*		

Index Correction . .

2)

Obsd. Mean Altitude
Refraction . . .
Semi-diam. . . .
Par. in Altitude . .

True Altitude (h). .
Latitude (ϕ) . . .
Polar Dist. (p) . .

2)

$s =$

° ′ ″

s . . . . log cos
$s - h$ . . . . log sin
$s - \phi$ . . . . log cosec
$s - p$ . . . . log sec

2)

∴ log tan $\frac{t}{2} =$

° ′ ″

$\frac{t}{2} =$

t in arc =

h. m. s.

t in Time ± =

If before Noon from 24 0 0

L.A.T. of Observation
Equation of Time ±
L.M.T. of Observation

Required for S.T. Chron. only.
Sid. Equivts. . . . . . {hrs. mins. secs.
Sid. Interval from L.M.N. . . =
Sid. Time of L.M.N. . . . . . =
L.S.T. of Observation . . . .

Chronometer Time of Observation . .

Chronometer—Slow or Fast on L.S.T. or L.M.T. . . . . } ±

Refraction.

Refcn. due to Altitude . .
Corr. for Bar. .
Corr. for Therm.
Corr. Refcn. .

Parallax.

Hor. Par. . .
×
Cos h . . .
Par. in Alt. . .

Polar Distance (p) and Dec. (δ).

(p is reckoned from the elevated pole)

° ′ ″

δ at G.M.N. . .

Hourly Var. × Interval in hrs. from G.M.N. .

δ at Mean of Obsn.

90 0 0

Polar Dist. p . .

(If the Latitude ϕ and Dec. δ be of the same name or sign, then δ is *subtracted from* 90°, but if of different name, it is *added to* 90°.

Interval from G.M.N.

h. m. s.

* Mean of Obsd. Times . . .

Approx. Chron. Correction if known . . . } ±

L.M.T. or L.S.T. of Observation

(If L.S.T. then to reduce to L.M.T.)

L.S.T. of L.M.N.

Sid. Intl. from L.M.N. . .

M.T. Equivts. {hrs. mins. secs.

L.M.T. of Observation . .

Corr. for Long. E. (−) W. (+)

G.M.T. of Mean of Observation

Hence,

Interval in Hrs. from G.M.N. .

Form V.—Azimuth by Sun or Star with a Theodolite.

Place................................ Lat........................ R.O...........................

Date................................. Long....................... Mag. bearing...................

Object Observed (Sun or Star) .... Bar........................ Therm.. ....

Reduction of Elements.

Refraction.		Parallax.	
Refraction due to Altitude .		Hor. Par. .	
Corr. for Bar. .		×	
Corr. for Therm.		Cos Altitude.	
Corrd. Refractn.		Par. in Alt. .	

Computation.

	° ′ ″
Mean Observed Altitude. . . .	
Corrections for Refraction . . .	
Corrections for Semi-diam. . . .	
Corrections for Parallax	
True Altitude (h) . . .	
Latitude (ϕ)	
Polar Distance (p) . . .	
	2)
s	
$s - p$	
$s - \phi$	
$s - h$	
∴ log sec. s	
log sec. $(s - p)$	
log sin $(s - \phi)$	
log sin $(s - h)$	
	2)
$\log \tan \frac{A}{2} =$	
in arc $\frac{A}{2} =$	2)
∴ A =	
Mean Angle between R.O. and Mean of the Observations to Sun or Star . . . . . . =	
∴ Azimuth of R.O. from *Elevated* Pole . . . =	

With Sun Observation only.

For S.T. Chron.

G.S.T. of G.M.N. . . .	
Correction for Long. at 9·86 secs. per hour (+ W. − E.) . . .	
L.S.T. of L.M.N. . .	
Sid. Time of Observation	
Sid. Intl. from L.M.N. .	
M.T. Equivts. hrs. . .	
M.T. Equivts. mins. .	
M.T. Equivts. secs. . .	
(from field book if with M.T. Chron.) L.M.T. of Obsn. .	
Correction for Long. E. or W. .	
G.M.T. of Observation . . .	
Interval from G.M.N. in hours .	

Declination of Sun.

	° ′ ″
Dec. δ at G.M.N.	
Hourly Var. × Interval . . .	
δ at Time of Observation .	
	90 0 0
Polar Dist. p or $90° - (\pm \delta)$	

N.B.—(p) is reckoned from the Elevated Pole.

∴ If ϕ and δ be of different names, δ will be negative and $p = 90 + \delta$.

In this formula, Lat. ϕ must be taken with positive sign, whether N. or S.

APPENDIX E.

Table I.

MEAN REFRACTION.

Barometer 30 ins. Thermometer 50° F.

Appt. Altitude °	′	Refraction ′	″
8	0	6	33.3
	5	6	29.6
	10	6	25.9
	15	6	22.3
	20	6	18.8
	25	6	15.3
	30	6	11.9
	35	6	8.5
	40	6	5.2
	45	6	2.0
	50	5	58.9
	55	5	55.7
9	0	5	52.6
	5	5	49.6
	10	5	46.6
	15	5	43.6
	20	5	40.7
	25	5	37.9
	30	5	35.1
	35	5	32.4
	40	5	29.6
	45	5	27.0
	50	5	24.3
	55	5	21.7
10	0	5	19.2
	5	5	16.7
	10	5	14.2
	15	5	11.7
	20	5	9.3
	25	5	6.9
	30	5	4.6
	35	5	2.3
	40	5	0.0
	45	4	57.8
	50	4	55.6
	55	4	53.4
11	0	4	51.3
	5	4	49.1
	10	4	47.0
	15	4	44.9
	20	4	42.9
	25	4	40.9
	30	4	38.9
	35	4	36.9
	40	4	35.0
	45	4	33.1
	50	4	31.2
	55	4	29.4
12	0	4	27.6
	5	4	25.7
	10	4	23.9
	15	4	22.2
	20	4	20.4
	25	4	18.7
	30	4	17.0
	35	4	15.3
	40	4	13.6
	45	4	12.0
	50	4	10.4
	55	4	8.8
13	0	4	7.2
	5	4	5.6
	10	4	4.1
	15	4	2.6
	20	4	1.0
	25	3	59.5
	30	3	58.1
	35	3	56.6
	40	3	55.2
	45	3	53.7
	50	3	52.3
	55	3	50.9
14	0	3	49.5
	5	3	48.1
	10	3	46.8
	15	3	45.5
	20	3	44.2
	25	3	42.9
	30	3	41.6
	35	3	40.3
	40	3	39.0
	45	3	37.7
	50	3	36.5
	55	3	35.3
15	0	3	34.1

Appt. Altitude °	′	Refraction ′	″
15	0	3	34.1
	5	3	32.9
	10	3	31.7
	15	3	30.5
	20	3	29.4
	25	3	28.2
	30	3	27.1
	35	3	25.9
	40	3	24.8
	45	3	23.7
	50	3	22.6
	55	3	21.5
16	0	3	20.5
	5	3	19.4
	10	3	18.4
	15	3	17.3
	20	3	16.3
	25	3	15.2
	30	3	14.2
	35	3	13.2
	40	3	12.2
	45	3	11.2
	50	3	10.3
	55	3	9.3
17	0	3	8.3
	5	3	7.3
	10	3	6.4
	15	3	5.5
	20	3	4.6
	25	3	3.7
	30	3	2.8
	35	3	1.9
	40	3	1.0
	45	3	0.1
	50	2	59.2
	55	2	58.3
18	0	2	57.5
	5	2	56.6
	10	2	55.8
	15	2	54.9
	20	2	54.1
	25	2	53.3
	30	2	52.4
	35	2	51.6
	40	2	50.8
	45	2	50.0
	50	2	49.2
	55	2	48.4
19	0	2	47.7
	5	2	46.9
	10	2	46.1
	15	2	45.3
	20	2	44.6
	25	2	43.9
	30	2	43.1
	35	2	42.3
	40	2	41.6
	45	2	40.9
	50	2	40.2
	55	2	39.5
20	0	2	38.8
	5	2	38.1
	10	2	37.4
	15	2	36.7
	20	2	36.0
	25	2	35.3
	30	2	34.6
	35	2	33.9
	40	2	33.3
	45	2	32.6
	50	2	32.0
	55	2	31.3
21	0	2	30.7
	5	2	30.0
	0	2	29.4
	15	2	28.7
	20	2	28.1
	25	2	27.5
	30	2	26.9
	35	2	26.3
	40	2	25.7
	45	2	25.1
	50	2	24.5
	55	2	23.9
22	0	2	23.3

Appt. Altitude °	′	Refraction ′	″
22	0	2	23.3
	10	2	22.1
	20	2	20.9
	30	2	19.8
	40	2	18.7
	50	2	17.5
23	0	2	16.4
	10	2	15.4
	20	2	14.3
	30	2	13.3
	40	2	12.3
	50	2	11.3
24	0	2	10.3
	10	2	9.2
	20	2	8.3
	30	2	7.3
	40	2	6.3
	50	2	5.3
25	0	2	4.4
	10	2	3.4
	20	2	2.5
	30	2	1.6
	40	2	0.7
	50	1	59.8
26	0	1	58.9
	10	1	58.1
	20	1	57.2
	30	1	56.4
	40	1	55.5
	50	1	54.7
27	0	1	53.9
	10	1	53.1
	20	1	52.3
	30	1	51.5
	40	1	50.7
	50	1	50.0
28	0	1	49.2
	10	1	48.4
	20	1	47.7
	30	1	46.9
	40	1	46.2
	50	1	45.5
29	0	1	44.8
	10	1	44.1
	20	1	43.4
	30	1	42.7
	40	1	42.0
	50	1	41.3
30	0	1	40.6
	10	1	39.9
	20	1	39.3
	30	1	38.6
	40	1	38.0
	50	1	37.3
31	0	1	36.7
	10	1	36.1
	20	1	35.5
	30	1	34.8
	40	1	34.2
	50	1	33.6
32	0	1	33.0
	10	1	32.4
	20	1	31.8
	30	1	31.3
	40	1	30.7
	50	1	30.1
33	0	1	29.5
	10	1	28.9
	20	1	28.4
	30	1	27.8
	40	1	27.3
	50	1	26.7
34	0	1	26.2
	10	1	25.6
	20	1	25.1
	30	1	24.6
	40	1	24.1
	50	1	23.6
35	0	1	23.1
	10	1	22.6
	20	1	22.0
	30	1	21.5
	40	1	21.0
	50	1	20.5
36	0	1	20.1

Appt. Altitude °	′	Refraction ′	″
36	0	1	20.1
	20	1	19.1
	40	1	18.2
37	0	1	17.3
	20	1	16.3
	40	1	15.4
38	0	1	14.5
	20	1	13.6
	40	1	12.7
39	0	1	11.9
	20	1	11.0
	40	1	10.2
40	0	1	9.3
	20	1	8.5
	40	1	7.8
41	0	1	7.0
	20	1	6.2
	40	1	5.4
42	0	1	4.7
	20	1	3.9
	40	1	3.2
43	0	1	2.4
	20	1	1.7
	40	1	1.0
44	0	1	0.3
	20	0	59.6
	40	0	58.9
45	0	0	58.2
	20	0	57.6
	40	0	56.9
46	0	0	56.3
	20	0	55.6
	40	0	55.0
47	0	0	54.3
	20	0	53.7
	40	0	53.1
48	0	0	52.5
	20	0	51.9
	40	0	51.3
49	0	0	50.6
	20	0	50.0
	40	0	49.6
50	0	0	48.9
	20	0	48.3
	40	0	47.7
51	0	0	47.3
	20	0	46.6
	40	0	46.0
52	0	0	45.3
	20	0	44.9
	40	0	44.4
53	0	0	43.9
	20	0	43.3
	40	0	42.8
54	0	0	42.3
	20	0	41.8
	40	0	41.3
55	0	0	40.8
	20	0	40.3
	40	0	39.8
56	0	0	39.3
	20	0	38.8
	40	0	38.3
57	0	0	37.8
	20	0	37.3
	40	0	36.8
58	0	0	36.4
	20	0	35.9
	40	0	35.4
59	0	0	35.0
	20	0	34.5
	40	0	34.0
60	0	0	33.6
	20	0	33.1
	40	0	32.7
61	0	0	32.3
	20	0	31.8
	40	0	31.4
62	0	0	30.9
	20	0	30.5
	40	0	30.1
63	0	0	29.7
	20	0	29.3
	40	0	28.8
64	0	0	28.4

Appt. Altitude °	′	Refraction ″
64	0	28.4
	30	27.8
65	0	27.2
	30	26.5
66	0	25.9
	30	25.3
67	0	24.7
	30	24.1
68	0	23.5
	30	22.9
69	0	22.4
	30	21.8
70	0	21.2
	30	20.6
71	0	20.1
	30	19.5
72	0	18.9
	30	18.3
73	0	17.8
	30	17.2
74	0	16.7
	30	16.1
75	0	15.6
	30	15.0
76	0	14.5
	30	14.0
77	0	13.5
	30	12.9
78	0	12.4
	30	11.9
79	0	11.3
	30	10.8
80	0	10.3
	30	9.7
81	0	9.2
	30	8.7
82	0	8.2
	30	7.7
83	0	7.2
	30	6.6
84	0	6.1
	30	5.6
85	0	5.1
	30	4.6
86	0	4.1
	30	3.6
87	0	3.1
	30	2.5
88	0	2.0
	30	1.5
89	0	1.0
	30	0.5
90	0	0.0

Table II.

CORRECTION OF THE MEAN REFRACTION FOR HEIGHT OF BAROMETER.

Below 30 ins. Subtract. Barometer	MEAN REFRACTION 0' 0"	0' 30"	1' 0"	1' 30"	2' 0"	2' 30"	3' 0"	3' 30"	4' 0"	4' 30"	5' 0"	5' 30"	6' 0"	6' 30"	Above 30 ins. Add. Barometer
	"	"	"	"	"	"	"	"	"	"	"	"	"	"	
23·00	0	7	14	21	30	36	48	49	54	63	70	77	86	91	
·2	0	7	14	20	27	34	41	46	54	61	68	75	81	88	
·4	0	7	13	20	26	33	40	46	53	60	66	73	79	86	
·6	0	7	13	19	26	31	38	45	51	57	64	70	77	83	
·8	0	6	12	19	26	31	37	43	50	56	63	68	74	81	
24·00	0	6	12	19	24	30	36	43	48	54	60	66	72	78	
·2	0	6	12	17	23	29	35	41	46	53	58	64	69	75	
·4	0	6	11	17	23	29	34	39	45	50	56	62	67	73	
·6	0	5	11	16	22	27	33	38	43	49	54	60	65	70	
·8	0	5	10	16	21	26	31	36	42	47	53	57	63	68	
25·00	0	5	10	15	20	25	30	35	40	45	50	55	60	65	
·2	0	5	10	16	19	24	29	34	38	43	48	53	57	62	
·4	0	5	9	14	18	23	28	32	37	41	46	51	55	60	
·6	0	4	9	13	18	22	26	31	30	39	44	49	53	57	
·8	0	4	8	13	17	21	25	29	34	38	42	46	50	55	
26·00	0	4	8	12	16	20	24	28	32	36	40	44	48	52	
·2	0	4	8	11	15	19	23	27	30	34	38	42	46	50	
·4	0	4	7	11	14	18	22	26	29	32	36	40	43	47	
·6	0	3	7	10	14	17	20	24	27	31	34	37	41	44	
·8	0	3	6	10	13	16	19	22	26	29	32	35	38	42	
27·00	0	3	6	9	12	15	18	21	24	27	30	33	36	39	
·2	0	3	6	9	11	14	17	20	23	26	28	31	34	37	
·4	0	3	5	8	10	13	15	18	21	23	26	28	31	34	
27·50	0	2	5	7	10	12	15	17	20	22	25	27	30	32	
·55	0	2	5	7	10	12	15	17	20	22	24	27	29	32	
·60	0	2	5	7	10	12	14	17	19	22	24	27	29	31	
·65	0	2	5	7	9	12	14	16	19	21	24	26	28	31	
·70	0	2	5	7	9	11	14	16	18	21	23	25	28	30	
27·75	0	2	4	7	9	11	13	16	18	20	23	25	27	29	
·80	0	2	4	7	9	11	13	15	18	20	22	24	27	29	
·85	0	2	4	6	9	11	13	15	17	19	22	24	26	28	
·90	0	2	4	6	8	10	13	15	17	19	21	23	25	27	
·95	0	2	4	6	8	10	12	14	16	18	21	23	25	27	
28·00	0	2	4	6	8	10	12	14	16	18	20	22	24	26	
·05	0	2	4	6	8	10	12	14	16	18	20	22	24	26	
·10	0	2	4	6	8	9	11	13	15	17	19	21	23	25	
·15	0	2	4	6	7	9	11	13	15	17	19	20	23	24	
·20	0	2	4	5	7	9	11	13	14	16	18	20	22	24	
28·25	0	2	3	5	7	9	10	12	14	16	18	19	21	23	
·30	0	2	3	5	7	8	10	12	14	16	17	19	21	22	
·35	0	2	3	5	7	8	10	12	13	15	17	18	20	22	
·40	0	2	3	5	6	8	10	11	13	14	16	18	19	21	
·45	0	2	3	5	6	8	9	11	12	14	16	17	18	20	
28·50	0	1	3	4	6	7	8	10	12	14	15	17	18	20	31·50
·55	0	1	3	4	6	7	9	10	12	13	14	16	17	19	·45
·60	0	1	3	4	6	7	8	10	11	13	14	15	17	18	·40
·65	0	1	3	4	5	7	8	9	11	12	14	15	16	18	·35
·70	0	1	3	4	5	6	8	9	10	12	13	14	16	17	·30
28·75	0	1	2	4	5	6	7	9	10	11	12	14	15	16	31·25
·80	0	1	2	4	5	6	7	8	10	11	12	13	14	16	·20
·85	0	1	2	3	5	6	7	8	9	10	12	13	14	15	·15
·90	0	1	2	3	4	5	7	8	8	10	11	12	13	16	·10
·95	0	1	2	3	5	5	6	7	8	9	11	13	13	14	·05
29·00	0	1	2	3	4	5	6	7	8	9	10	11	12	13	31·00
·05	0	1	2	3	4	5	6	7	8	9	10	11	11	12	30·95
·10	0	1	2	3	4	4	5	6	7	8	9	10	11	12	·90
·15	0	1	2	3	3	4	5	6	7	8	9	9	10	11	·85
·20	0	1	2	2	3	4	5	6	7	7	8	9	10	10	·80
29·25	0	1	1	2	3	4	4	5	6	7	8	8	9	10	30·75
·30	0	1	1	2	3	3	4	5	6	6	7	8	8	9	·70
·35	0	1	1	2	3	3	4	5	5	6	7	7	8	9	·65
·40	0	1	1	2	2	3	4	4	5	5	6	7	7	8	·60
·45	0	1	1	2	2	3	3	4	4	5	6	6	7	7	·55
29·50	0	0	1	1	2	2	3	3	4	5	5	6	6	7	30·50
·55	0	0	1	1	2	2	3	3	4	4	5	5	6	6	·45
·60	0	0	1	1	2	2	2	3	3	4	4	4	5	5	·40
·65	0	0	1	1	1	2	2	2	2	3	4	4	4	4	·35
·70	0	0	1	1	1	1	2	2	2	3	3	3	4	4	·30
29·75	0	0	0	1	1	1	1	2	2	2	3	3	3	3	30·25
·80	0	0	0	1	1	1	1	1	2	2	2	2	2	3	·20
·85	0	0	0	0	1	1	1	1	1	1	2	2	2	2	·15
·90	0	0	0	0	0	0	1	1	1	1	1	1	1	1	·10
·95	0	0	0	0	0	0	0	0	0	0	1	1	1	1	·05
30·00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30·00
Subtract.	0' 0"	0' 30"	1' 0"	1' 30"	2' 0"	2' 30"	3' 0"	3' 30"	4' 0"	4' 30"	5' 0"	5' 30"	6' 0"	6' 30" MEAN REFRACTION	Add.

NOTE.—Should the Barometer stand at *less* than 23 inches the Barometer correction can be computed from the following quantities, which are made out for each inch of the Barometer below 30 inches, and which are dependent on the altitudes.

Altitude	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	30°	35°	40°	45°	50°	55°	60°	70°	80°
Bar. corr. for each 1" below 30"	10·6"	9·8"	9·0"	8·3"	7·7"	7·18"	6·73"	6·31"	5·96"	5·61"	5·31"	5·04"	4·79"	4·57"	4·36"	4·16"	3·97"	3·81"	3·65"	3·50"	3·36"	2·76"	2·32"	1·94"	1·60"	1·31"	1·12"	0·71"	0·36"

Table III.

CORRECTION OF THE MEAN REFRACTION FOR HEIGHT OF THERMOMETER.

THERMO. ADD.	MEAN REFRACTION.																THERMO. ADD.
	0′		1′		2′		3′		4′		5′		6′		7′		
	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	
°	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	°
− 10	0	4	8	12	16	20	24	28	33	37	41	46	50	55	60	65	− 10
− 8	0	4	8	12	15	19	23	27	31	36	40	44	48	53	58	62	− 8
− 6	0	4	7	11	15	19	22	26	30	34	38	42	47	51	55	60	− 6
− 4	0	4	7	11	14	18	22	25	29	33	37	41	45	49	53	57	− 4
− 2	0	3	7	10	14	17	21	24	28	31	35	39	43	47	51	55	− 2
0	0	3	7	10	13	16	20	23	27	30	34	37	41	45	49	53	0
+ 2	0	3	6	9	12	16	19	22	25	29	32	36	39	43	47	50	+ 2
4	0	3	6	9	12	15	18	21	24	28	31	34	37	41	44	48	4
6	0	3	6	8	11	14	17	20	23	26	29	32	36	39	42	46	6
8	0	3	5	8	11	14	16	10	22	25	28	31	34	37	40	43	8
10	0	3	5	8	10	13	15	18	21	24	26	29	32	35	38	41	10
11	0	2	5	7	10	13	15	18	20	23	26	28	31	34	37	40	11
12	0	2	5	7	10	12	15	17	20	22	25	28	30	33	36	39	12
13	0	2	5	7	9	12	14	17	19	22	24	27	30	32	35	38	13
14	0	2	5	7	9	11	14	16	19	21	24	26	29	31	34	37	14
15	0	2	4	7	9	11	13	16	18	20	23	25	28	30	33	36	15
16	0	2	4	6	9	11	13	15	18	20	22	25	27	29	32	35	16
17	0	2	4	6	8	10	13	15	17	19	21	24	26	29	31	33	17
18	0	2	4	6	8	10	12	14	16	19	21	23	25	28	30	32	18
19	0	2	4	6	8	10	12	14	16	18	20	22	24	27	29	31	19
20	0	2	4	6	8	9	11	13	15	17	19	22	24	26	28	30	20
21	0	2	4	5	7	9	11	13	15	17	19	21	23	25	27	29	21
22	0	2	3	5	7	9	11	12	14	16	18	20	22	24	26	28	22
23	0	2	3	5	7	8	10	12	14	15	17	19	21	23	25	27	23
24	0	2	3	5	6	8	10	11	13	15	17	18	20	22	24	26	24
25	0	2	3	5	6	8	9	11	13	14	16	18	19	21	23	25	25
26	0	1	3	4	6	7	9	11	12	14	15	17	19	20	22	24	26
27	0	1	3	4	6	7	9	10	12	13	15	16	18	19	21	23	27
28	0	1	3	4	5	7	8	10	11	12	14	15	17	19	20	22	28
29	0	1	3	4	5	6	8	9	11	12	13	15	16	18	19	21	29
30	0	1	2	4	5	6	7	9	10	11	13	14	15	17	18	20	30
31	0	1	2	3	5	6	7	8	9	11	12	13	15	16	17	19	31
32	0	1	2	3	4	6	7	8	9	10	11	13	14	15	16	18	32
33	0	1	2	3	4	5	6	7	8	10	11	12	13	14	15	17	33
34	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	34
35	0	1	2	3	4	5	6	6	7	8	9	10	11	13	14	15	35
36	0	1	2	3	3	4	5	6	7	8	9	10	11	12	13	14	36
37	0	1	2	2	3	4	5	6	6	7	8	9	10	11	12	13	37
38	0	1	1	2	3	4	4	5	6	7	7	8	9	10	11	12	38
39	0	1	1	2	3	3	4	5	5	6	7	8	8	9	10	11	39
40	0	1	1	2	2	3	4	4	5	6	6	7	8	8	9	10	40
41	0	1	1	2	2	3	3	4	4	5	6	6	7	7	8	9	41
42	0	0	1	1	2	2	3	3	4	4	5	5	6	7	7	8	42
43	0	0	1	1	2	2	3	3	3	4	4	5	5	6	6	7	43
44	0	0	1	1	1	2	2	3	3	3	4	4	4	5	5	6	44
45	0	0	1	1	1	1	2	2	2	3	2	3	4	4	4	5	45
46	0	0	0	1	1	1	1	2	2	2	2	2	3	3	4	4	46
47	0	0	0	1	1	1	1	1	1	2	2	2	2	2	3	3	47
48	0	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	48
49	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	49
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	50
	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	
ADD.	0′		1′		2′		3′		4′		5′		6′		7′		ADD
THERMO.	MEAN REFRACTION.																THERMO.

Table III.—*continued.*

CORRECTION OF THE MEAN REFRACTION FOR HEIGHT OF THERMOMETER.

THERMO. SUB-TRACT	MEAN REFRACTION.																THERMO. SUB-TRACT
	0′		1′		2′		3′		4′		5′		6′		7′		
	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	
°	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	″	°
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	50
51	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	51
52	0	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	52
53	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2	3	53
54	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	54
55	0	0	1	1	1	1	2	2	2	3	3	3	4	4	4	5	55
56	0	0	1	1	1	2	2	2	3	3	4	4	4	5	5	6	56
57	0	0	1	1	2	2	2	3	3	4	4	5	5	6	6	6	57
58	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	58
59	0	1	1	2	2	3	3	4	4	5	5	6	6	7	8	8	59
60	0	1	1	2	2	3	3	4	5	5	6	7	7	8	9	9	60
61	0	1	1	2	3	3	4	4	5	6	7	7	8	9	9	10	61
62	0	1	1	2	3	3	4	5	6	6	7	8	9	9	10	11	62
63	0	1	1	2	3	4	5	5	6	7	8	8	9	10	11	12	63
64	0	1	2	2	3	4	5	6	7	7	8	9	10	11	12	13	64
65	0	1	2	3	3	4	5	6	7	8	9	10	11	12	13	14	65
66	0	1	2	3	4	5	6	6	7	8	9	10	11	12	14	15	66
67	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	67
68	0	1	2	3	4	5	6	7	8	9	11	11	13	14	15	16	68
69	0	1	2	3	4	5	7	8	9	10	11	12	13	15	16	17	69
70	0	1	2	3	5	6	7	8	9	10	12	12	14	16	17	18	70
71	0	1	2	4	5	6	7	8	10	11	12	13	15	16	18	19	71
72	0	1	2	4	5	6	8	9	10	11	13	14	16	17	18	20	72
73	0	1	3	4	5	7	8	9	11	12	13	14	16	18	19	21	73
74	0	1	3	4	5	7	8	10	11	12	14	15	17	18	20	22	74
75	0	1	3	4	6	7	8	10	11	13	14	16	18	19	21	22	75
76	0	1	3	4	6	7	9	10	12	13	15	16	18	20	22	23	76
77	0	1	3	5	6	8	9	11	12	14	16	17	19	21	22	24	77
78	0	2	3	5	6	8	9	11	13	14	16	18	20	21	23	25	78
79	0	2	3	5	6	8	10	11	13	15	17	18	20	22	24	26	79
80	0	2	3	5	7	8	10	12	14	15	17	19	21	23	25	27	80
81	0	2	3	5	7	9	10	12	14	16	18	20	21	24	26	28	81
82	0	2	4	5	7	9	11	13	14	16	18	20	22	24	26	28	82
83	0	2	4	5	7	9	11	13	15	17	19	21	23	25	27	29	83
84	0	2	4	6	8	9	11	13	15	17	19	21	23	26	28	30	84
85	0	2	4	6	8	10	12	14	16	18	20	22	24	26	29	31	85
86	0	2	4	6	8	10	12	14	16	18	20	23	25	27	29	32	86
87	0	2	4	6	8	10	12	14	17	19	21	23	25	28	30	32	87
88	0	2	4	6	8	10	13	15	17	19	21	24	26	28	31	33	88
89	0	2	4	6	9	11	13	15	17	20	22	24	27	29	32	34	89
90	0	2	4	7	9	11	13	16	18	20	23	25	27	30	32	35	90
91	0	2	4	7	9	11	14	16	18	21	23	25	28	31	33	36	91
92	0	2	5	7	9	11	14	16	19	21	24	26	29	31	34	37	92
93	0	2	5	7	9	12	14	17	19	22	24	27	29	32	35	37	93
94	0	2	5	7	10	12	14	17	19	22	25	27	30	33	35	38	94
95	0	2	5	7	10	12	15	17	20	22	25	28	30	33	36	39	95
96	0	2	5	7	10	12	15	18	20	23	26	28	31	34	37	40	96
97	0	3	5	8	10	13	15	18	21	23	26	29	32	35	38	41	97
98	0	3	5	8	10	13	16	18	21	24	27	29	32	35	38	41	98
99	0	3	5	8	11	13	16	19	21	24	27	30	33	36	39	42	99
100	0	3	5	8	11	13	16	19	22	25	28	31	34	37	40	43	100
	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	0″	30″	
SUB-TRACT	0′		1′		2′		3′		4′		5′		6′		7′		SUB-TRACT
THERMO.	MEAN REFRACTION.																THERMO.

Table IV.

SUN'S PARALLAX IN ALTITUDE.

Date.	Altitude.										Date.
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	
	"	"	"	"	"	"	"	"	"	"	
January 1st ...	9·00	8·86	8·46	7·79	6·89	5·79	4·50	3·08	1·57	0	
February 1st ..	8·98	8·85	8·44	7·78	6·88	5·77	4·49	3·07	1·56	0	December 1st.
March 1st ...	8·92	8·79	8·38	7·72	6·83	5·74	4·46	3·05	1·55	0	November 1st.
April 1st ..	8·85	8·72	8·32	7·66	6·78	5·69	4·42	3·03	1·54	0	October 1st.
May 1st ...	8·78	8·65	8·25	7·60	6·73	5·65	4·39	3·00	1·53	0	September 1st
June 1st ..	8·72	8·59	8·20	7·55	6·68	5·61	4·36	2·98	1·52	0	August 1st.
July 1st ...	8·70	8·57	8·18	7·53	6·66	5·59	4·35	2·97	1·51	0	

Example of Correction for Refraction and Parallax.

On December 10th the observed altitude of the Sun was 25° 10′ 25″. Barometer 29 73. Thermometer 41° F.

From Table I.	Correction for Altitude	=	2′ 3·4″
† II.	„ Barometer	=	− 1
† III.	„ Thermometer	=	+ 2
	Correction for Refraction	=	2′ 4·4″
IV.	Correction for Parallax	=	8·1″

Observed Altitude	=	25° 10′ 25″
Correction for Refraction	=	− 2 4·4
„ Parallax	=	+ 8·1
True Altitude	=	25° 8′ 28·7″

† These two corrections are found with argument mean refraction.

Table V.

REDUCTION OF CIRCUM-MERIDIAN ALTITUDES TO THE MERIDIAN

$$m = \frac{2\sin^2\frac{1}{2}t}{\sin 1''}$$

t	m.	1m.	2m.	3m.	4m.
s.	″	″	″	″	″
0	0·0	2·0	7·9	17·7	31·4
1	0·0	2·0	8·0	17·9	31·7
2	0·0	2·1	8·1	18·1	31·9
3	0·0	2·2	8·2	18·3	32·2
4	0·0	2·3	8·4	18·5	32·5
5	0·0	2·3	8·5	18·7	32·7
6	0·0	2·4	8·7	18·9	33·0
7	0·0	2·4	8·8	19·1	33·3
8	0·0	2·5	8·9	19·3	33·5
9	0·0	2·6	9·1	19·5	33·8
10	0·1	2·7	9·2	19·7	34·1
11	0·1	2·7	9·4	19·9	34·4
12	0·1	2·8	9·5	20·1	34·6
13	0·1	2·9	9·6	20·3	34·9
14	0·1	3·0	9·8	20·5	35·2
15	0·1	3·1	9·9	20·7	35·5
16	0·1	3·1	10·1	20·9	35·7
17	0·2	3·2	10·2	21·2	36·0
18	0·2	3·3	10·4	21·4	36·3
19	0·2	3·4	10·5	21·6	36·6
20	0·2	3·5	10·7	21·8	36·9
21	0·2	3·6	10·8	22·0	37·2
22	0·3	3·7	11·0	22·3	37·4
23	0·3	3·8	11·2	22·5	37·7
24	0·3	3·8	11·3	22·7	38·0
25	0·3	3·9	11·5	22·9	38·3
26	0·4	4·0	11·6	23·1	38·6
27	0·4	4·1	11·8	23·4	38·9
28	0·4	4·2	11·9	23·6	39·2
29	0·5	4·3	12·1	23·8	39·5
30	0·5	4·4	12·3	24·0	39·8
31	0·5	4·5	12·4	24·3	40·1
32	0·6	4·6	12·6	24·5	40·3
33	0·6	4·7	12·8	24·7	40·6
34	0·6	4·8	12·9	25·0	40·9
35	0·7	4·9	13·1	25·2	41·2
36	0·7	5·0	13·3	25·4	41·5
37	0·7	5·1	13·4	25·7	41·8
38	0·8	5·2	13·6	25·9	42·1
39	0·8	5·3	13·8	26·2	42·5
40	0·9	5·4	14·0	26·4	42·8
41	0·9	5·6	14·1	26·6	43·1
42	1·0	5·7	14·3	26·9	43·4
43	1·0	5·8	14·5	27·1	43·7
44	1·1	5·9	14·7	27·4	44·0
45	1·1	6·0	14·8	27·6	44·3
46	1·2	6·1	15·0	27·9	44·6
47	1·2	6·2	15·2	28·1	44·9
48	1·3	6·4	15·4	28·3	45·2
49	1·3	6·5	15·6	28·6	45·5
50	1·4	6·6	15·8	28·8	45·9
51	1·4	6·7	15·9	29·1	46·2
52	1·5	6·8	16·1	29·4	46·5
53	1·5	7·0	16·3	29·6	46·8
54	1·6	7·1	16·5	29·9	47·1
55	1·6	7·2	16·7	30·1	47·5
56	1·7	7·3	16·9	30·4	47·8
57	1·8	7·5	17·1	30·6	48·1
58	1·8	7·6	17·3	30·9	48·4
59	1·9	7·7	17·5	31·2	48·8

t	5m.	6m.	7m.	8m.	9m.
s.	″	″	″	″	″
0	49·1	70·7	96·2	125·7	159·0
1	49·4	71·1	96·7	126·2	159·6
2	49·7	71·5	97·1	126·7	160·2
3	50·1	71·9	97·6	127·2	160·8
4	50·4	72·3	98·0	127·8	161·4
5	50·7	72·7	98·5	128·3	162·0
6	51·1	73·1	99·0	128·8	162·6
7	51·4	73·5	99·4	129·3	163·2
8	51·7	73·9	99·9	129·9	163·8
9	52·1	74·3	100·4	130·4	164·4
10	52·4	74·7	100·8	130·9	165·0
11	52·7	75·1	101·3	131·5	165·6
12	53·1	75·5	101·8	132·0	166·2
13	53·4	75·9	102·3	132·6	166·8
14	53·8	76·3	102·7	133·1	167·4
15	54·1	76·7	103·2	133·6	168·0
16	54·5	77·1	103·7	134·2	168·6
17	54·8	77·5	104·2	134·7	169·2
18	55·1	77·9	104·6	135·3	169·8
19	55·5	78·3	105·1	135·8	170·4
20	55·8	78·8	105·6	136·3	171·0
21	56·2	79·2	106·1	136·9	171·6
22	56·5	79·6	106·6	137·4	172·2
23	56·9	80·0	107·0	138·0	172·9
24	57·3	80·4	107·5	138·5	173·5
25	57·6	80·8	108·0	139·1	174·1
26	58·0	81·3	108·5	139·6	174·7
27	58·3	81·7	109·0	140·3	175·3
28	58·7	82·1	109·5	140·7	175·9
29	59·0	82·5	110·0	141·3	176·6
30	59·4	83·0	110·4	141·8	177·2
31	59·8	83·4	110·9	142·4	177·8
32	60·1	83·8	111·4	143·0	178·4
33	60·5	84·2	111·9	143·5	179·0
34	60·8	84·7	112·4	144·1	179·7
35	61·2	85·1	112·9	144·6	180·3
36	61·6	85·5	113·4	145·2	180·9
37	61·9	86·0	113·9	145·8	181·6
38	62·3	86·4	114·4	146·3	182·2
39	62·7	86·8	114·9	146·9	182·8
40	63·0	87·3	115·4	147·5	183·5
41	63·4	87·7	115·9	148·0	184·1
42	63·8	88·1	116·4	148·6	184·7
43	64·2	88·6	116·9	149·2	185·4
44	64·5	89·0	117·4	149·7	186·0
45	64·9	89·5	117·9	150·3	186·6
46	65·3	89·9	118·4	150·9	187·3
47	65·7	90·3	118·9	151·5	187·9
48	66·0	90·8	119·5	152·0	188·5
49	66·4	91·2	120·0	152·6	189·2
50	66·8	91·7	120·5	153·2	189·8
51	67·2	92·1	121·0	153·8	190·5
52	67·6	92·6	121·5	154·3	191·1
53	68·0	93·0	122·0	154·9	191·8
54	68·3	93·5	122·5	155·5	192·4
55	68·7	93·9	123·1	156·1	193·1
56	69·1	94·4	123·6	156·7	193·7
57	69·5	94·8	124·1	157·3	194·4
58	69·9	95·3	124·6	157·8	195·0
59	70·3	95·7	125·1	158·4	195·7

t	10m.	11m.	12m.	13m.	14m.
s.	″	″	″	″	″
0	196·3	237·5	282·7	331·7	384·7
1	197·0	238·3	283·5	332·6	385·6
2	197·6	239·0	284·3	333·4	386·6
3	198·3	239·7	285·0	334·3	387·5
4	198·9	240·4	285·8	335·2	388·4
5	199·6	241·1	286·6	336·0	389·3
6	200·3	241·9	287·4	336·8	390·2
7	200·9	242·6	288·2	337·7	391·2
8	201·6	243·3	289·0	338·6	392·1
9	202·2	244·1	289·8	339·4	393·0
10	202·9	244·8	290·6	340·3	393·9
11	203·6	245·5	291·4	341·2	394·9
12	204·2	246·3	292·2	342·0	395·8
13	204·9	247·0	293·0	342·9	396·7
14	205·6	247·7	293·8	343·7	397·6
15	206·3	248·5	294·6	344·6	398·6
16	206·9	249·2	295·4	345·5	399·5
17	207·6	249·9	296·2	346·4	400·5
18	208·3	250·7	297·0	347·2	401·4
19	208·9	251·4	297·8	348·1	402·3
20	209·6	252·2	298·6	349·0	403·3
21	210·3	252·9	299·4	349·8	404·2
22	211·0	253·6	300·2	350·7	405·1
23	211·7	254·4	301·0	351·6	406·1
24	212·3	255·1	301·8	352·5	407·0
25	213·0	255·9	302·6	353·3	408·0
26	213·7	256·6	303·5	354·2	408·9
27	214·4	257·4	304·3	355·1	409·8
28	215·1	258·1	305·1	356·0	410·8
29	215·8	258·9	305·9	356·9	411·7
30	216·4	259·6	306·7	357·7	412·7
31	217·1	260·4	307·5	358·6	413·6
32	217·8	261·1	308·4	359·5	414·6
33	218·5	261·9	309·2	360·4	415·5
34	219·2	262·6	310·0	361·3	416·5
35	219·9	263·4	310·8	362·2	417·4
36	220·6	264·1	311·6	363·1	418·4
37	221·3	264·9	312·5	364·0	419·4
38	222·0	265·7	313·3	364·8	420·3
39	222·7	266·4	314·1	365·7	421·3
40	223·4	267·2	315·0	366·6	422·2
41	224·1	268·0	315·8	367·5	423·2
42	224·8	268·7	316·6	368·4	424·1
43	225·5	269·5	317·4	369·3	425·1
44	226·2	270·2	318·3	370·2	426·1
45	226·9	271·0	319·1	371·1	427·0
46	227·6	271·8	319·9	372·0	428·0
47	228·3	272·6	320·8	372·9	429·0
48	229·0	273·3	321·6	373·8	429·9
49	229·7	274·1	322·4	374·7	430·9
50	230·4	274·9	323·3	375·6	431·9
51	231·1	275·6	324·1	376·5	432·8
52	231·8	276·4	325·0	377·4	433·8
53	232·5	277·2	325·8	378·3	434·8
54	233·2	278·0	326·7	379·2	435·8
55	234·0	278·8	327·5	380·2	436·7
56	234·7	279·5	328·4	381·1	437·7
57	235·4	280·3	329·2	382·0	438·7
58	236·1	281·1	330·0	382·9	439·7
59	236·8	281·9	330·9	383·8	440·6

t	15m.	16m.	17m.	18m.	19m.
s.	″	″	″	″	″
0	441·6	502·5	567·2	635·9	708·4
1	442·6	503·5	568·3	637·0	709·7
2	443·6	504·6	569·4	638·2	710·9
3	444·6	505·6	570·5	639·4	712·1
4	445·6	506·7	571·6	640·6	713·4
5	446·5	507·7	572·8	641·7	714·6
6	447·5	508·8	573·9	642·9	715·9
7	448·5	509·8	575·0	644·1	717·1
8	449·5	510·9	576·1	645·3	718·4
9	450·5	511·9	577·2	646·5	719·6
10	451·5	513·0	578·4	647·7	720·9
11	452·5	514·0	579·5	648·9	722·1
12	453·5	515·1	580·6	650·0	723·4
13	454·5	516·1	581·7	651·2	724·6
14	455·5	517·2	582·9	652·4	725·9
15	456·5	518·3	584·0	653·6	727·2
16	457·5	519·3	585·1	654·8	728·4
17	458·5	520·4	586·2	656·0	729·7
18	459·5	521·5	587·4	657·2	730·9
19	460·5	522·5	588·5	658·4	732·2
20	461·5	523·6	589·6	659·6	733·5
21	462·5	524·7	590·8	660·8	734·7
22	463·5	525·7	591·9	662·0	736·0
23	464·5	526·8	593·0	663·2	737·3
24	465·5	527·9	594·2	664·4	738·5
25	466·5	529·0	595·3	665·6	739·8
26	467·5	530·0	596·5	666·8	741·1
27	468·5	531·1	597·6	668·0	742·3
28	469·5	532·2	598·7	669·2	743·6
29	470·5	533·3	599·9	670·4	744·9
30	471·6	534·3	601·0	671·6	746·2
31	472·6	535·4	602·2	672·8	747·4
32	473·6	536·5	603·3	674·1	748·7
33	474·6	537·6	604·5	675·3	750·0
34	475·6	538·7	605·6	676·5	751·3
35	476·6	539·7	606·8	677·7	752·6
36	477·6	540·8	607·9	678·9	753·8
37	478·7	541·9	609·1	680·1	755·1
38	479·7	543·0	610·2	681·3	756·4
39	480·7	544·1	611·4	682·6	757·7
40	481·7	545·2	612·5	683·8	759·0
41	482·8	546·3	613·7	685·0	760·2
42	483·8	547·4	614·8	686·2	761·5
43	484·8	548·4	616·0	687·4	762·8
44	485·8	549·5	617·2	688·7	764·1
45	486·9	550·6	618·3	689·9	765·4
46	487·9	551·7	619·5	691·1	766·7
47	488·9	552·8	620·6	692·4	768·0
48	490·0	553·9	621·8	693·6	769·3
49	491·0	555·0	623·0	694·8	770·6
50	492·0	556·1	624·1	696·0	771·9
51	493·1	557·2	625·3	697·3	773·1
52	494·1	558·3	626·5	698·5	774·5
53	495·2	559·4	627·6	699·7	775·7
54	496·2	560·5	628·8	701·0	777·1
55	497·2	561·6	630·0	702·2	778·4
56	498·3	562·8	631·2	703·5	779·7
57	499·3	563·9	632·3	704·7	781·0
58	500·4	565·0	633·5	705·9	782·3
59	501·4	566·1	634·7	707·1	783·6

t	20m.	21m.	22m.	23m.	24m.
s.	″	″	″	″	″
0	784·9	865·3	949·6	1037·8	1129·9
1	786·2	866·7	951·0	1039·3	1131·5
2	787·5	868·0	952·5	1040·8	1133·1
3	788·8	869·4	953·9	1042·3	1134·6
4	790·1	870·8	955·4	1043·8	1136·2
5	791·4	872·2	956·8	1045·3	1137·8
6	792·7	873·5	958·2	1046·8	1139·4
7	794·0	874·9	959·7	1048·3	1140·9
8	795·4	876·3	961·1	1049·9	1142·5
9	796·7	877·7	962·6	1051·4	1144·1
10	798·0	879·1	964·0	1052·9	1145·7
11	799·3	880·4	965·5	1054·4	1147·2
12	800·7	881·8	966·9	1055·9	1148·8
13	802·0	883·2	968·4	1057·4	1150·4
14	803·3	884·6	969·8	1058·9	1152·0
15	804·6	886·0	971·3	1060·5	1153·6
16	806·0	887·4	972·7	1062·0	1155·2
17	807·3	888·8	974·2	1063·5	1156·7
18	808·6	890·2	975·6	1065·0	1158·3
19	809·9	891·6	977·1	1066·6	1159·9
20	811·3	893·0	978·6	1068·1	1161·5
21	812·6	894·4	980·0	1069·6	1163·1
22	813·9	895·7	981·5	1071·1	1164·7
23	815·2	897·1	982·9	1072·7	1166·3
24	816·6	898·5	984·4	1074·2	1167·9
25	817·9	900·0	985·9	1075·7	1169·5
26	819·2	901·4	987·3	1077·2	1171·1
27	820·5	902·8	988·8	1078·8	1172·7
28	821·9	904·2	990·3	1080·3	1174·2
29	823·3	905·6	991·7	1081·8	1175·8
30	824·6	907·0	993·2	1083·4	1177·4
31	825·9	908·4	994·7	1084·9	1179·1
32	827·3	909·8	996·2	1086·4	1180·6
33	828·6	911·3	997·6	1087·9	1182·2
34	829·9	912·6	999·1	1089·5	1183·9
35	831·3	914·0	1000·6	1091·0	1185·5
36	832·6	915·5	1002·1	1092·6	1187·1
37	834·0	916·9	1003·5	1094·1	1188·7
38	835·3	918·3	1005·0	1095·7	1190·3
39	836·7	919·7	1006·5	1097·2	1191·9
40	838·0	921·1	1008·0	1098·8	1193·5
41	839·4	922·5	1009·4	1100·3	1195·1
42	840·7	923·9	1010·9	1101·9	1196·7
43	842·1	925·3	1012·4	1103·4	1198·3
44	843·4	926·8	1013·9	1105·0	1199·9
45	844·8	928·2	1015·4	1106·5	1201·5
46	846·2	929·6	1016·9	1108·1	1203·1
47	847·5	931·0	1018·4	1109·6	1204·7
48	848·9	932·4	1019·9	1111·2	1206·4
49	850·3	933·8	1021·4	1112·7	1208·0
50	851·6	935·3	1022·8	1114·3	1209·6
51	852·9	936·7	1024·3	1115·8	1211·2
52	854·3	938·1	1025·8	1117·4	1212·9
53	855·7	939·5	1027·3	1119·0	1214·5
54	857·1	940·9	1028·8	1120·5	1216·1
55	858·4	942·4	1030·3	1122·1	1217·7
56	859·8	943·9	1031·8	1123·6	1219·4
57	861·2	945·3	1033·3	1125·2	1221·0
58	862·5	946·7	1034·8	1126·8	1222·6
59	863·9	948·1	1036·3	1128·3	1224·3

t	25m.	26m.	27m.	28m.	29m.
s.	″	″	″	″	″
0	1225·9	1325·9	1429·7	1537·5	1649·0
1	1227·5	1327·6	1431·4	1539·3	1650·9
2	1229·2	1329·3	1433·2	1541·1	1652·8
3	1230·8	1331·0	1434·9	1542·9	1654·7
4	1232·5	1332·7	1436·7	1544·8	1656·6
5	1234·1	1334·4	1438·5	1546·6	1658·5
6	1235·7	1336·1	1440·3	1548·4	1660·4
7	1237·3	1337·8	1442·1	1550·3	1662·3
8	1239·0	1339·5	1443·9	1552·1	1664·2
9	1240·6	1341·2	1445·6	1553·9	1666·1
10	1242·3	1342·9	1447·4	1555·8	1668·0
11	1243·9	1344·6	1449·2	1557·6	1669·9
12	1245·6	1346·3	1451·0	1559·5	1671·9
13	1247·2	1348·0	1452·8	1561·3	1673·8
14	1248·9	1349·7	1454·5	1563·2	1675·7
15	1250·6	1351·4	1456·3	1565·0	1677·6
16	1252·2	1353·2	1458·1	1566·9	1679·5
17	1253·9	1354·9	1459·9	1568·7	1681·4
18	1255·5	1356·6	1461·6	1570·5	1683·3
19	1257·1	1358·3	1463·4	1572·4	1685·2
20	1258·8	1360·1	1465·2	1574·2	1687·2
21	1260·5	1361·8	1466·9	1576·1	1689·1
22	1262·2	1363·5	1468·7	1577·9	1691·0
23	1263·8	1365·2	1470·5	1579·8	1692·9
24	1265·5	1367·0	1472·3	1581·7	1694·8
25	1267·1	1368·7	1474·1	1583·5	1696·7
26	1268·8	1370·4	1475·9	1585·3	1698·6
27	1270·5	1372·1	1477·7	1587·2	1700·5
28	1272·1	1373·9	1479·5	1589·1	1702·5
29	1273·7	1375·6	1481·3	1590·9	1704·4
30	1275·4	1377·3	1483·1	1592·8	1706·3
31	1277·1	1379·0	1484·9	1594·6	1708·2
32	1278·8	1380·8	1486·7	1596·5	1710·2
33	1280·4	1382·5	1488·5	1598·3	1712·1
34	1282·1	1384·2	1490·3	1600·2	1714·0
35	1283·8	1385·9	1492·1	1602·1	1715·9
36	1285·5	1387·7	1493·9	1604·0	1717·9
37	1287·1	1389·4	1495·7	1605·8	1719·8
38	1288·8	1391·2	1497·5	1607·7	1721·7
39	1290·5	1392·9	1499·3	1609·6	1723·6
40	1292·2	1394·7	1501·1	1611·5	1725·6
41	1293·8	1396·4	1502·9	1613·3	1727·5
42	1295·5	1398·2	1504·7	1615·2	1729·5
43	1297·2	1399·9	1506·5	1617·1	1731·5
44	1298·9	1401·7	1508·4	1618·9	1733·4
45	1300·5	1403·4	1510·2	1620·8	1735·3
46	1302·2	1405·2	1512·0	1622·7	1737·2
47	1303·9	1406·9	1513·8	1624·6	1739·2
48	1305·6	1408·7	1515·6	1626·5	1741·2
49	1307·3	1410·4	1517·4	1628·3	1743·1
50	1309·0	1412·2	1519·2	1630·2	1745·1
51	1310·7	1413·9	1521·0	1632·1	1747·0
52	1312·4	1415·7	1522·9	1634·0	1749·0
53	1314·1	1417·4	1524·7	1635·9	1750·9
54	1315·7	1419·2	1526·5	1637·7	1752·8
55	1317·4	1420·9	1528·3	1639·6	1754·8
56	1319·1	1422·7	1530·2	1641·5	1756·8
57	1320·8	1424·4	1532·0	1643·4	1758·7
58	1322·5	1426·2	1533·8	1645·2	1760·7
59	1324·2	1427·9	1535·6	1647·1	1762·6

APPENDIX F.

Mr. Wilfrid Airy's Method of Determining Heights and Distances.

The simplest form of tacheometer is an ordinary theodolite, without additions of any kind, used in conjunction with a staff on which a known distance is marked by two well-defined lines. Angles of elevation or depression 'from the horizontal' are observed to the marks on the staff, held vertically at the point observed to. If α and β be the vertical angle of the *upper* and *lower* lines respectively, whilst l is the distance apart of these lines, and L or H the horizontal or vertical distances respectively, of the *lower* line, then, $L = \dfrac{l}{\tan\alpha - \tan\beta}$ and $H = \dfrac{l\tan\beta}{\tan\alpha - \tan\beta}$, when both α and β are angles of elevation. When both are angles of depression $\tan\alpha - \tan\beta$ becomes $\tan\beta - \tan\alpha$, whilst when α alone is an angle of elevation, the denominator is $\tan\alpha + \tan\beta$. The required factor can readily be worked out by the use of a table of 'natural tangents.'

This is an excellent and simple method of determining heights and distances, and capable of producing fairly accurate results, dependent, obviously, on the minuteness of the subdivisions of a degree which can be read on the vertical arc of the theodolite.

If the staff be held in a horizontal position and perpendicular to a line passing through one of the marks and the theodolite, the angles required may be read off the horizontal limb, but in this way, distances only can be determined.

The following formulæ, due to Mr. T. N. Roberts of Cape Town, are more convenient for logarithmic calculations:—

A and B. Both readings either above or below the horizontal, staff held vertically

$$v = \frac{h\cos\alpha\cos\beta}{\sin(\alpha - \beta)},\ x = y\tan\beta.$$

C. One reading above and the other below the horizontal, staff held vertically

$$y = \frac{h\cos\alpha\cos\beta}{\sin(\alpha + \beta)},\ x = y\tan\beta.$$

In this case it is immaterial whether α is greater or less than β.

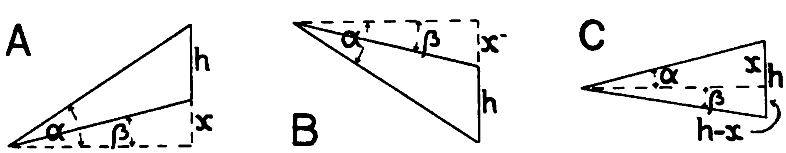


Fig. 125a.

APPENDIX G.

Points and Crossings.

Points or switches are the mechanical appliances (fig. 126) by means of which a vehicle or a train is maintained on one line of rails at a junction, or is diverted on to another line of rails.

Switch, & stock rail.

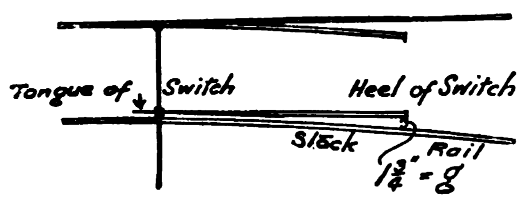


Fig. 126.

A crossing (fig. 127), separated from the switch by a portion of straight or curved rail, is the arrangement which permits of a wheel and flange passing across one or more rails which go to constitute one or more of the lines of a railway.

The straight or curve which connects the point or switch with the crossing can be set out in the manner already described under the head of railway curves,

Crossing.

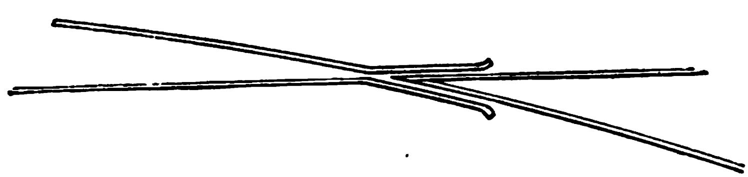


Fig. 127.

but, as the points and switches are usually made in standard lengths, the tongue of the switch is not situated at the point of tangency of the curve, but usually some

distance from it in the direction of the crossing. The distance from the switch to the crossing is measured from the heel of the switch and not from the tongue. There are also other fixed dimensions which affect the position of the curve. It is, therefore desirable that the setting out of 'points and crossings' should be considered.

For the purposes of illustration the following dimensions are assumed, and vary but little in different lines of English standard gauge:—

Gauge of railway	4′ 8½″
Gauge between lines of railway	6′ 0″
Length of switch in main line . .	12′ 0″, 15′ 0″ or 18′ 0″
Length of switch in sidings	6′ 0″ to 9′ 0″
Width of top table of rail	2½″
Minimum clearance between heel of switch and stock rail	1¾″

SINGLE JUNCTION.

The simplest form of junction is where a single branch line leaves one which is straight (fig. 128).

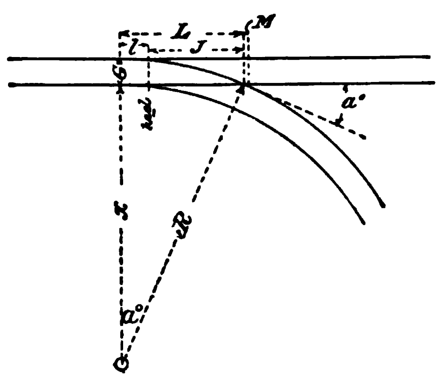


FIG. 128.

A radius commonly employed for such a junction when used as a siding or crossover road is 602 feet.

R = Radius of curve of outside rail.

G = Gauge of railway.

a° = Angle of crossing.

$x = R - G$.

L = Length from tangent point of curve to centre of crossing.

l = Length from tangent point to heel of switch.

Then

$$L = \sqrt{(R + x)\,G} = 75'\ 1 \cdot 7''.$$

$$\operatorname{Sin} a^\circ = \frac{L}{R}$$

$$R = \frac{G}{\operatorname{versin} a^\circ}$$

The distance from the inside of the stock rail to the inside of the switch rail at heel of switch $= g = 4\frac{1}{4}''$, therefore in this case the length l, the distance from the tangent of the curve to the heel of the switch $= \sqrt{(R + R - g)g} = 20'\ 7\cdot 8''$, and the length J, the distance from the heel of the switch to the point of the crossing $= L - l = 54'\ 5\cdot 9''$.

There is still further correction to be made for the point of the crossing, which in the above calculation is assumed to be infinitely fine, but in practice the width of the nose of the crossing is made $= \frac{1}{2}''$. Thus, with a 1 in 8 crossing, which is produced by the intersection of the straight line by one of 602 feet radius, 4″ will have to be added to the length J, which becomes 54′ 10″ nearly, in the illustration.

Nothing appears to be gained by making the length of the switch in feet greater than twice the ratio of the crossing to unity. Thus, in the case under consideration, the switch will be 15 feet long, and its tongue or point will be 5′ 7″·8 from the tangent point.

Curve Running from Reverse Curve.

When a junction curve runs out of another curve of the same or some other radius in a reverse direction (fig. 129).

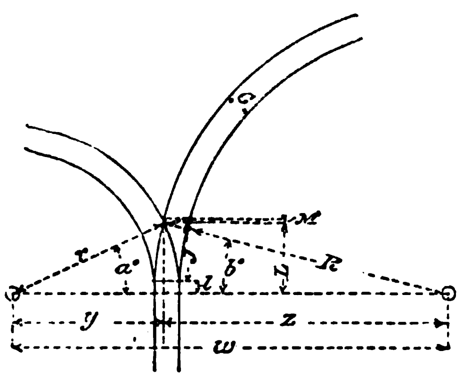


Fig. 129.

Let R = Radius of the inside rail of main curve.

r = Radius of the outside rail of branch curve.

G = Gauge of railway.

L = Length from tangent point of curve to centre of crossing.

w = Distance from centre to centre of radii.

M = Distance from point of crossing to nose as made.

Then

$$w = z + y = (R + r) - G$$

and
$$z^2 - y^2 = R^2 - r^2$$

or
$$(z + y)(z - y) = R^2 - r^2$$

$$\therefore z - y = \frac{R^2 - r^2}{w} = 0 \text{ if } R = r$$

i.e. $w\,(z - y) = R^2 - r^2$

and $z = w - y = y$ if $R = r$

From these equations we can find z and y.

Then $$L = \sqrt{(r^2 - y^2)} = \sqrt{(R^2 - z^2)}$$

$$\text{Sin } a^\circ = \frac{L}{r} \qquad \text{Sin } b^\circ = \frac{L}{R}$$

$$\text{Angle of crossing} = a^\circ + b^\circ$$

The distance l from the tangent point to heel of switch may be found in the same way, substituting $g = 4\frac{1}{2}''$ in the instance quoted for G in a standard gauge railway.

Curve Running from Inside of Curve.

When a junction curve runs out of another curve of greater radius in the same direction (fig. 130).

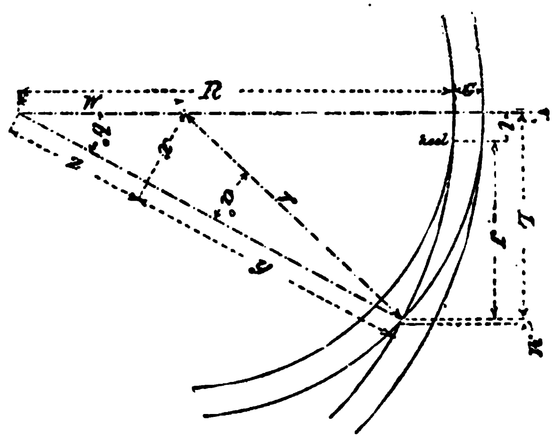


FIG. 130.

$$W = G + (R - r)$$

$$(y - z) = \frac{r^2 - W^2}{R} \quad \text{for} \quad r^2 - W^2 = y^2 - z^2 = (y + z)\,(y - z)$$
$$= (y - z)\,R$$

and $y = R - z$, whence we can find y and z.

$$x = \sqrt{r^2 - y^2} \quad \text{and} \quad L - M = R \;\; \sin \;\; b^\circ = \frac{R\,x}{W}$$

$$\text{Sin } a^\circ = \frac{x}{r}$$

If W be greater than r, then

$$(z - y) = \frac{W^2 - r^2}{R}$$

$$y = R - z$$

Double Junctions.

Double junctions (fig. 131) may be laid out by following the above rules, which are equally applicable to three, four or more lines of railway.

In the case of obtuse crossings, as *n* and *n'* (fig. 131), the addition M to the length from the heel of the switch to the point of the crossing is not applicable.

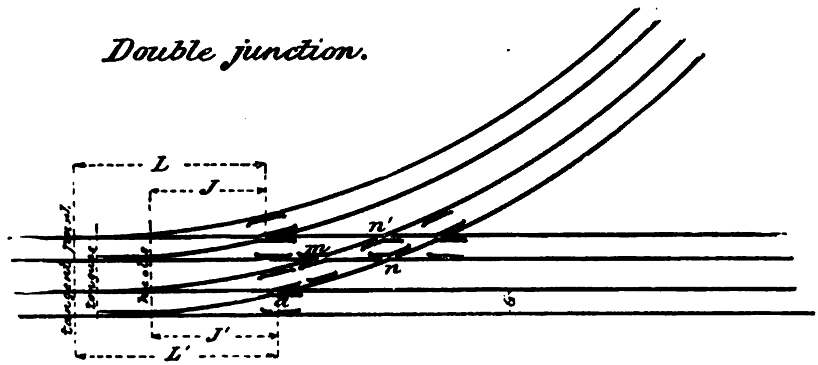


FIG. 131.

Acute crossings, as at *a* (fig. 131), are seldom made of a smaller spread than 1 in 18, which represents the intersection of a straight line by a curve of 3049 feet, or 46 chains radius nearly.

APPENDIX H.

Sun Observations.

The main points of difference between observations of the sun and those of the fixed stars are:—

1. That whereas the right ascensions and declinations of the fixed stars are practically constant—and hence do not change during twenty-four hours, or during the time occupied by a set of observations—those of the sun, moon and planets may change appreciably during the same interval.

Hence, if an observation of the sun be made, it is necessary to find the corresponding *Greenwich time*, in order that the declination of the sun at that instant may be obtained from the Nautical Almanac for purposes of computation.

The maximum rate of change of declination—that is, at the vernal and autumnal equinoxes—is, however, only about one minute per hour, so that, if we know correct local time within five or ten minutes and the longitude within, say, a degree of arc or four minutes of time, the Greenwich mean time of the observation can be found within the degree of accuracy required for determining the correct declination nearly enough when the observations are made with an ordinary instrument.

An error of fifteen minutes in the computed Greenwich time will mean, at the most, about fifteen seconds in the declination, and few ordinary instruments can be relied upon to give so small an error.

At the solstices—midsummer and midwinter—the rate of change is much slower, and a larger error in the Greenwich time makes no appreciable difference.

The movements of the moon are, however, much more rapid. The declination of the moon may change as much as fifteen minutes per hour, and consequently, the data for computing Greenwich time must be known with considerable accuracy to make lunar observations of any value.

For this, and other reasons mentioned in the text, little or nothing has been said about lunar observations.

The declination being known, the same formulæ can be applied to the sun as to the fixed stars.

Inversion of Image.

Secondly, a fixed star being for practical purposes, a mere point, it can be exactly bisected by the cross hairs, and no complication is introduced by the inversion of the image; which takes place in every inverting telescope, such as that of the ordinary theodolite.

But the sun's disc being of very appreciable magnitude—rather more than half a degree—it is impossible to bisect it with certainty, and hence the observa-

tions are taken to the edges of the disc, as shown in fig. 132, in which K L and M N represent the cross hairs and A D E B the sun's image, as seen through the telescope.

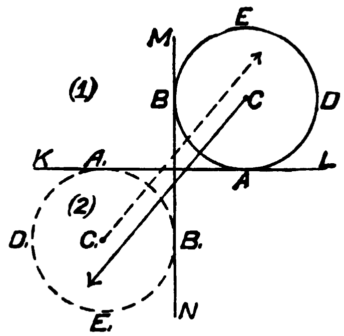


FIG. 132.

It is usually necessary to get both the vertical and horizontal hairs tangent to the disc, as shown.

The Nautical Almanac gives the angular value of the sun's semi-diameter—which varies with the distance of the sun from the earth—for every day of the year. Thus, if the altitude of the edge A be read off, that of the centre C can be found by applying the correction for semi-diameter.

The inversion of the image by the telescope must now be taken into account, however. With the position of the image shown the correction is to be *subtract d*, not added, inasmuch as the inversion in a vertical direction takes place about the plane K L, and hence the real disc is *below* that plane, and the point A, which is the *lower* edge or limb of the image, is the *upper* limb of the true sun.

Similar reasoning applies to horizontal angles, and the true position of the sun to give the image A D E B would be that shown by the dotted circle. Hence, to get the azimuth of the sun's centre from that of the edge B with a theodolite graduated from left to right, as usual, the correction for semi-diameter must be *added*, not subtracted, as would appear from the figure. If the image were in either of the quadrants (1) or (2) in the figure the correction would have to be deducted in azimuth, and this whether the sun be rising or setting.

Recording Observations.

It is evident that some regular system of recording such observations must be followed to avoid misunderstandings as to which limb was actually observed. In the Field Book, in Appendix B, they are marked to be recorded as seen through the telescope. This may tend to prevent errors in the field, particularly with beginners.

It is perhaps more usual and more convenient, however, to record the readings as referring to the true sun. Thus, in the figure the altitude of the point A would be recorded as that of the sun's *upper* limb, and the azimuth of B as that of the *right* limb.

Observing both Limbs.

It is more satisfactory to observe both limbs, so as to eliminate the correction for semi-diameter.

In the northern hemisphere, outside the tropics, we shall always be looking in a southerly direction when observing the sun.

Hence the east, whence the sun is rising, will be on the left, and if we watch him through the telescope when rising he will appear to move somewhat in the direction shown by the full arrow in fig. 132, his true path being that shown by the dotted arrow. If setting, in the northern hemisphere, he will appear to move somewhat as shown by the full arrow in fig. 134, p. 305, his true path being that shown by the dotted arrow.

To *follow* him with the telescope, however, keeping the image always in the

same position relatively to the hairs, it would of course be necessary to move the telescope in the direction of the true path.

To observe both limbs, then, it is convenient, having made contact with both hairs in one quadrant, to allow one hair to remain fixed until the sun, by his own movement, makes contact with it on the other side of it, following him with the second wire only.

Thus, suppose the sun rising in the northern hemisphere, and that we decide to let the horizontal hair remain fixed.

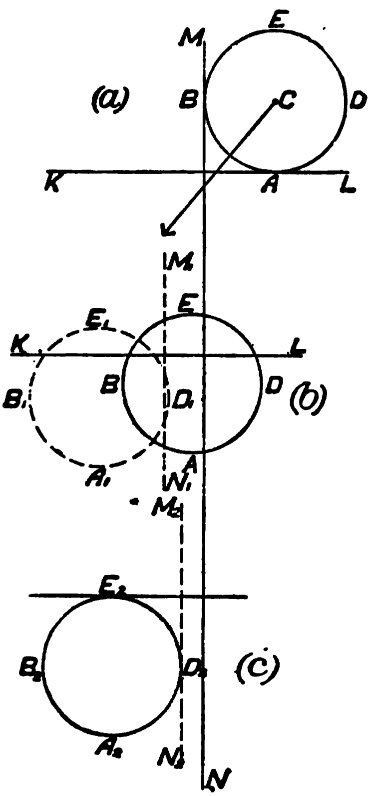


FIG. 133.

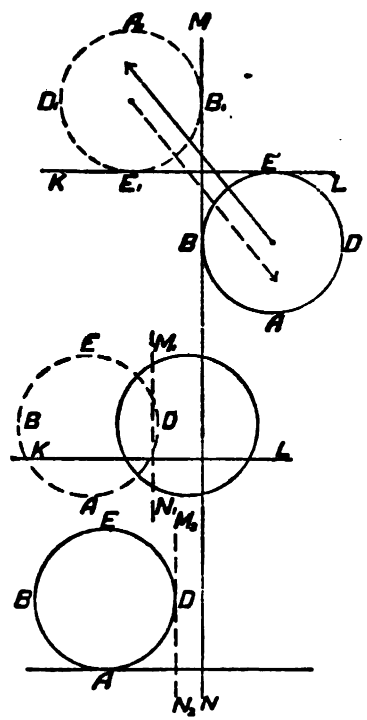


FIG. 134.

Make contact with both hairs, as at A B, fig. 133 (*a*), the true sun being below the hair. Read the time, and also the horizontal limb of the theodolite, to obtain the azimuth of the sun's right limb.

The altitude circle need not be read at this stage, as it will remain fixed.

By the time the readings are taken the image will be about in the position shown in fig. 133 (*b*), and as the image moves it is clear that the point E will ultimately make contact with the wire K L on the lower side.

But in order that the point D may make contact with M N at the same instant, the latter must be moved.

If we move it towards D, as seen in the image, the latter will mov further

away from it. It must be moved in the opposite direction until the hair and image occupy some such position as shown by the dotted lines.

Then carefully follow the disc with the hair M N by means of the horizontal tangent screw, keeping the hair always so near the edge that it is just visible, until the moment when contact is made with the horizontal wire. At that moment the vertical wire should be made tangential also (fig. 133, *c*).

The time and azimuth are again read, and also the altitude circle, if it has not been read previously.

The mean of the times and the mean of the azimuth readings will give the

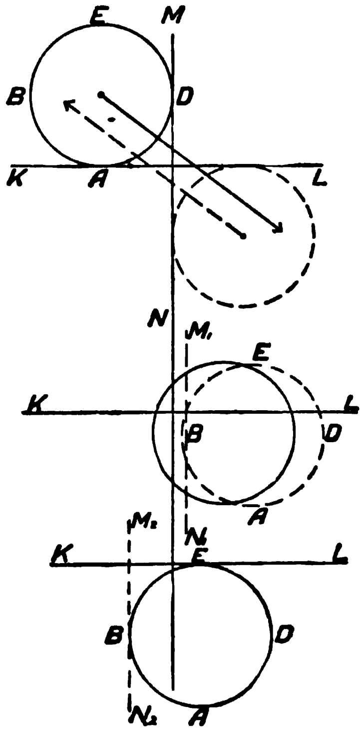


FIG. 135.

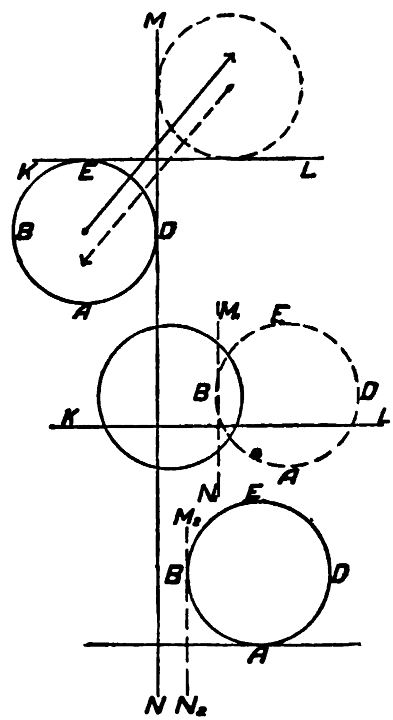


FIG. 136.

time and azimuth reading when the sun's centre was at the given altitude, and those means are used just as for a star observation after reducing the time to Greenwich time, so as to obtain the correct declination, and correcting for parallax, as described below, and for refraction and level error (p. 16), as in the case of a star.

In fig. 134 (*a*) (*b*) and (*c*), a series of positions with sun *setting* in the northern hemisphere are shown. In (*a*) the full and dotted circles show the image and the corresponding position of the true sun respectively, the full and dotted arrows showing the apparent and true directions of movement; (*b*) and (*c*) correspond to the similar views in fig. 133, and show later positions of the image only.

In fig. 135 a similar series of three views is shown, with sun rising in the

southern hemisphere, outside the tropics, while fig. 136 relates to the case where the sun is setting in the southern hemisphere or in the tropics with the sun north of zenith.

These observations should be repeated with reversed face, and the means of the four readings—one pair with face left and one with face right—will give the altitude and azimuth of the sun's centre and the time to be used in the calculations.

This eliminates the semi-diameter correction.

It is best to take two pairs of observations, as described, with the sun rising and two with the sun setting, at *about* the same altitude, and computing azimuth (or whatever may be required) from the morning and afternoon readings separately, take the mean result as correct.

The method of equal altitudes (as described on p. 163, for time) is uncertain and inconvenient, among other reasons because the instrument cannot well be moved, and independent observations are to be generally preferred.

Parallax.

The third difference between sun and star observations arises from the comparative proximity of the sun to the earth.

In consequence of this there is an appreciable difference between the lines drawn, from the sun's centre, to the centre of the earth and to a point of observation on its surface.

This difference clearly diminishes as the sun approaches the zenith, where the two lines will coincide.

This is shown in fig. 137, where E is supposed to represent the earth and S the sun.

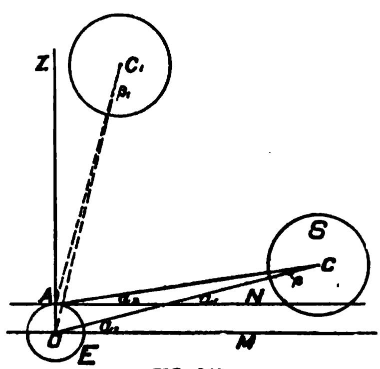


FIG. 137.

A being the point of observation, the true celestial horizon is O M, and the true altitude of the sun is M O C or a_1. But the natural horizon, as given by the theodolite, is A N, and the measured altitude is N A C, or a_2, which differs from a_1 by the angle A C O, or β. When the sun is near the zenith, Z, this approaches zero, as shown by the dotted lines.

The correction for parallax reduces the observations made on the surface to the centre of the earth. The value of the correction is given in Appendix E, p. 295, an average value being about 9 × cos altitude, in seconds.

Observations at Equal Altitudes.

If the object is to obtain time and azimuth readings at equal altitudes, morning and afternoon (as described on p. 163), it must be remembered that it is difficult to read the true lower limb during the afternoon. The true path of the sun being then as shown by the dotted arrow, its apparent path will be shown by the full arrow. Now, unless special means are provided for illuminating it, the hair K L, which is fixed, will be invisible until part of the

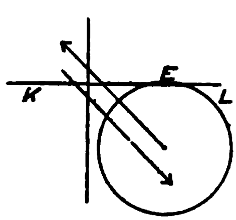


FIG. 138.

sun's disc has actually crossed it, and the moment of tangency at point E cannot be obtained.

Observing without Dark Glass.

In the event of the dark glass attached to the eye-piece being misplaced or broken, the following expedient may be adopted.

Fit a piece of stout cardboard accurately to the inside of the dew-cap. Mark the centre accurately, and pierce a small hole there; enlarge it by riming it out with a hard fine-pointed pencil.

This will sufficiently reduce the light to enable the sun to be observed without a dark glass. The size of the hole depends upon the telescope, and must be settled by trial. With this method it is almost essential to observe both limbs, on account of the errors due to aberration, which tend to alter the apparent semi-diameter of the sun.

Reduction of Circum-Meridian Altitudes.

If the observations are made on both sides of, and near to, the meridian, they may be reduced to the latter by the formulæ on p. 151 for computing latitude.

If in doubt as to the correct time, the observations may also be reduced to the meridian graphically with fair accuracy.

Take for example the following readings:

Altitude.			Time.			Altitude.			Time.		
°	′	″	h.	m.	s.	°	′	″	h.	m.	s.
34	56	8	23	49	0	34	57	11	23	57	43
34	56	31	23	50	23	34	57	3	23	58	46
34	56	53	23	51	50	34	56	48	0	0	13
34	57	6	23	52	51	34	56	26	0	1	45
34	57	18	23	55	57						

These are plotted in a curve in fig. 139, and, taking a smooth curve through the points, it is evident that the maximum, or meridian, altitude is about $34° 57′ 20″$.

The plotting of the results in a curve also serves to show whether they can be regarded as a good set of readings or not, this being decided by the smoothness of the curve.

From the meridian altitude and the declination the latitude is obtained by the precepts on p. 145.

Routine for Azimuth.

When the latitude is known, azimuth is best observed by independent altitudes taken in pairs, as described, upper and lower limb, with face left and face right, some distance from the meridian. The routine to be followed would be somewhat as follows.

Having set up the theodolite at the station, the other end of the line whose true azimuth is required is marked by a signal—the 'referring object'—which should be so far distant as to be visible with solar focus.

It is perhaps most convenient to set the theodolite either to zero or to the computed bearing of the line and direct it to the referring object by unclamping the lower plate. Then, unclamping the upper plate, take two pairs of readings of both limbs of the sun, face right and face left, as already described, and find the mean readings.

Having corrected for parallax and refraction and reduced the mean of the time readings to Greenwich time—unless they were read direct from a chronometer set to true Greenwich time—find, from the Nautical Almanac, the declination of the sun at that instant.

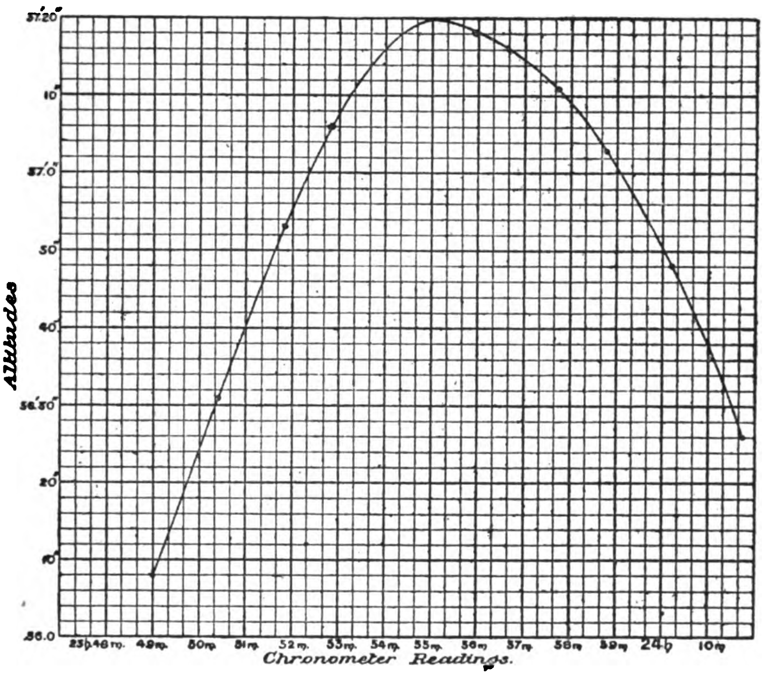


FIG. 139.

Then use the formula on p. 165, viz.—add together the mean of the observed altitudes, the latitude and the polar distance of the sun, which is the complement of its declination, and divide by 2.

Call the half sum s, and find its log. secant.

From s deduct the polar distance and take log. secant of result.

" s " latitude and take log. sin. of result.

" s " altitude " " "

Add together these logarithms, and the half sum will give the log. tan. of *half* the true azimuth of the sun at the mean of the observed positions. Multiply by 2 and we get the true azimuth. Let this angle $= \beta$. It will be the angle from the *elevated* pole to the mean position of the sun, measured in the nearest direction. The mean of the observed readings on the horizontal circle—supposing the theodolite set to zero at referring object—gives the angle from the referring object to the mean position of the sun, measured clockwise. Call this angle θ.

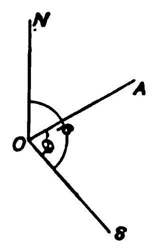


FIG. 140.

The angle from the elevated pole to the referring object can then be found by a mere process of addition or subtraction, and a little

thought will readily show whether the two are to be added or subtracted in any case.

Thus, in fig. 140, if A be the referring object, O N the meridian, S the mean observed position of the sun, $AOS = \theta$, and $NOS = \beta$. Then the required azimuth $NOA = \beta - \theta$.

If the south be the elevated pole, the angle must be corrected by adding 180°, the angle from N to S.

It is probably best to represent the positions in a figure, as above, in each case.

For an example, *vide* p. 166.

Computing Time as Check.

It is well to compute the time also, from the same observations, by the formulæ explained on pp. 159 and 160.

In the formula on p. 159, t gives the hour angle of the true sun, expressed in degrees, minutes and seconds at the mean instant of the observations. This, reduced to mean solar hours, minutes and seconds, gives the mean solar time interval from apparent noon. Then, by the equation of time (p. 136), the mean solar time interval from mean noon is found. This should correspond with the chronometer reading if the observations were in the afternoon, or, deducted from twelve, should give the mean chronometer reading for morning observations.

The hour angle may be reduced to a mean solar time interval by remembering that 15° hour angle = 1 hour in time.

If the chronometer reading is much out it may be necessary to recompute the azimuth with the new declination found from the corrected time. It is always useful to take a second set of observations, starting on the referring object with a different reading. This tends to eliminate graduation errors.

APPENDIX I.

Investigation of Formula for Levelling with the Barometer.

The method is based on the fact that the pressure of the air decreases as we ascend; firstly, because there is a less actual depth of air above the higher level, and also, because the density of the air decreases with the elevation.

Suppose A and B (fig. 141) to represent two stations at different levels. Let B M = X be the difference of level.

Let R be any point in A B whose height above A M = x, and let p = pressure of air per sq. ft. at R. Let Q be a point a little higher, so that the difference of level from R to Q = dx, and let dp be the corresponding change in pressure, which will be negative. This loss of pressure is due to, and equal to, the weight of a column of air one sq. ft. in the base, and whose height is equal to the difference of level dx.

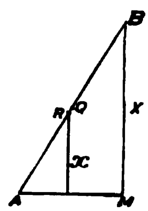


FIG. 141.

This latter being small the density of the air may be regarded as constant from R to Q. Denote the weight of 1 cub. ft. of the air by δ.

Then volume of column = $1 \times dx = dx$ cub. ft.

$\therefore$ weight of column of air = $\delta \times dx$ lbs.

But this = change in pressure = $-dp$.

$$\therefore \quad \delta \times dx = -dp \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We must now find δ, the weight of 1 cub. ft. of air at R. Let w = weight of 1 cub. ft. of air at a temperature of 32° F., and under a standard pressure P, and in the latitude of 45°.

Then, by the laws of the expansion of gases, if δ be the weight of a cub. ft. at any other temperature $\tau°$ F., and pressure p in the same latitude,

$$\frac{\delta}{w} = \frac{p}{P} \cdot \left(\frac{493}{493 + \tau - 32}\right)$$

$$\therefore \quad \delta = \frac{w}{P}\left(\frac{493}{493 + \tau - 32}\right) \cdot p \quad . \quad . \quad . \quad . \quad (2)$$

If the latitude change to ϕ the force of gravity changes also to $\dfrac{1}{1 + \cdot 002695 \cos 2\phi}$ of its value at 45° latitude.

Hence the weight of a cub. ft. of air at latitude ϕ, pressure p and temperature $\tau°$ F. becomes

$$\delta = \frac{w}{P}\left(\frac{493}{493 + \tau - 32}\right) \cdot p \left(\frac{1}{1 + \cdot 002695 \cos 2\phi}\right) \quad . \quad . \quad (3)$$

Substitute this in (1) and we get

$$\frac{w}{\mathrm{P}}\left(\frac{493}{493+\tau-32}\right)\left(\frac{1}{1+\cdot 002695 \cos 2\phi}\right) p \times dx = -dp$$

If we suppose τ constant from A to B we may write this

$$\mathrm{C}\, p\, dx = -dp$$

where C is constant and equal to

$$\frac{w}{\mathrm{P}}\left(\frac{493}{493+\tau-32}\right)\left(\frac{1}{1+\cdot 002695 \cos 2\phi}\right)$$

or

$$\mathrm{C}\, dx = -\frac{dp}{p}$$

Integrating,

$$\mathrm{C\,X} = -\int_{\mathrm{P}_1}^{\mathrm{P}_2} \frac{dp}{p}$$

$$= \log_e\left(\frac{\mathrm{P}_1}{\mathrm{P}_2}\right)$$

where P_1 and P_2 are the pressures at A and B respectively.

Hence $\mathrm{C\,X} = \log_e\left(\frac{\mathrm{P}_1}{\mathrm{P}_2}\right)$ if τ be constant. . . . . . . (4)

In practice τ is not constant, and hence a mean value is taken for it, viz. $\frac{\tau+\tau_1}{2}$ where τ and τ_1 are the air temperatures (as read by the detached thermometers) at A and B.

The pressures P_1 and P_2 should, it would appear at first sight, be expressed in the same units as the standard pressure P. But this is really immaterial, for so long as P_1 *and* P_2 are in the same units, say inches of mercury, and so long as the mercury is at the *same temperature* in both, their *ratio* will be the same, and hence they can be represented by the barometer readings β, β_1.

But if the temperatures of the mercury, as read on the attached thermometers, are different at the two stations, a correction must be introduced to allow for the expansion of the mercury and scale, whence we derive the corrective factor $\frac{1}{1+\cdot 0001\,(t-t')}$ on p. 200.

If we take P in lbs. per sq. ft. and w in lbs., and substitute proper values for them, as obtained by experiment, we shall arrive at a result.

Now, according to Professor Everett ('Units and Physical Constants'), 1 c.c. of air at 32° F. weighs ·0012759 grm. under a pressure of 10^6 dynes per sq. cm., and in the latitude of 45° the value of g is 980·61 cms. per sec. per sec.

Reducing the units, we obtain the result that one cub. ft. of air weighs $\frac{\cdot 0012759 \times (2\cdot 54)^3 \times 12^3}{453\cdot 59}$ lbs. under a pressure of $\frac{10^6 \times (2\cdot 54)^3 \times 12^3}{13825 \times 980\cdot 61}$ lbs. per sq. ft. in the latitude 45°.

Hence,

$$\frac{w}{\mathrm{P}} = \frac{\cdot 001275 \times 13825 \times 980\cdot 61}{453\cdot 9 \times 10}$$

Hence, putting $\frac{\tau + \tau_1}{2}$ instead of τ in the equation for C, we obtain

$$C = \frac{\cdot 001275 \times 13825 \times 980\cdot 61}{453\cdot 9 \times 10^6} \times \left(\frac{493}{493 + \frac{\tau + \tau_1}{2} - 32}\right)\left(\frac{1}{1 + \cdot 002695 \cos 2\phi}\right)$$

$$\therefore X = \frac{\log_e \frac{P_1}{P_2}}{C}$$

$$= \frac{453\cdot 9 \times 10^6}{\cdot 001275 \times 13825 \times 980\cdot 61} \times \frac{986 + \tau + \tau_1 - 64}{986}(1 + \cdot 002695 \cos 2\phi)$$

$$\times\ 2\cdot 3026 \times \log_{10}\left\{\frac{\beta}{\beta_1} \times \frac{1}{1 + \cdot 0001\,(t - t_1)}\right\}$$

$$= 60382 \times \{1 + \cdot 001014(\tau + \tau_1 - 64)\}\,(1 + \cdot 002695 \cos 2\phi) \times \log\left\{\frac{\beta}{\beta_1} \times \frac{1}{1 + \cdot 0001\,(t - t_1)}\right\}$$

which agrees nearly with the formula on p. 200.

The difference is due to the use of slightly different values for the constants.

The effect of the corrections for latitude and for the expansion of the scale is always very small.

The assumption that the mean temperature is $\frac{\tau + \tau_1}{2}$, on the contrary, may often lead to more or less serious errors.

The unavoidable error in finding the mean temperature of the air generally far exceeds that due to neglecting the corrections for latitude and for attached thermometers, and renders levelling by the barometer always uncertain.

It is best to take a number of readings of the air temperature at intermediate points, and so get a nearer approximation to the true mean.

If it is desired to correct for the expansion of the mercury and the scales it is perhaps more convenient, as well as more exact, to reduce each barometric reading separately to freezing point.

The Hypsometer.

It has been found, experimentally, that an increase of pressure of about $26\cdot 8$ mms. of mercury causes a difference of 1° C. in the boiling-point of water.

Hence, $\frac{5}{9} \times 26\cdot 8$ mms., or $\cdot 586$ inch of mercury will cause a difference of 1° F. if the change in boiling point be assumed proportional to that in pressure.

This is not quite true, and it is better to obtain from tables the pressures corresponding to the boiling points; but, on the above assumption, remembering that at $29\cdot 92$ ins. of mercury water boils at 212° F., we obtain the result that, if the boiling point differs from 212° by t, then the pressure will be $29\cdot 92 \pm t \times \cdot 586$, according as the boiling point is higher or lower than 212°.

Another formula, given by Cotterill, and more exact, is

$$\log p = 5 \cdot \frac{t - 212}{t + 367} + \log 14\cdot 7$$

where p is the pressure in lbs. per sq. in. and t is the Fahrenheit temperature.

The following table, from Cotterill's treatise on the 'Steam Engine,' gives the pressures corresponding to a few boiling points.

Boiling point (° Fahrenheit)	208	209	210	211	212	213	214	215	216
Pressure (lbs. per sq. in.)	13·57	13·84	14·12	14·41	14·70	14·99	15·29	15·60	15·91

APPENDIX J.

On Velocity of Approach in the Gauging of Streams by Notches.

The usual formula for the discharge over a rectangular notch, with an appreciable velocity of approach, as given by English writers, is

$$Q = \frac{2}{3} c\, l \sqrt{2g}\left\{(h + h_v)^{3/2} - h_v^{3/2}\right\}$$

where h_v is the extra head due to the velocity of approach, and $= \frac{v^2}{2g}$, where v is this velocity, other letters having the same significance as on p. 225.

If the velocity of approach can be determined by current meters, or Pitot tubes, or otherwise, the correct discharge can be readily approximated to by this formula. If, on the other hand, no means exist for determining the velocity of approach, an approximation to the discharge may still be obtained as follows:

1. Measure the sectional area of the stream at or about the point of observation.

2. Compute an approximate discharge in cub. ft. per sec. by the ordinary formulæ (p. 229), neglecting the velocity of approach.

3. Divide this approximate discharge by the sectional area in sq. ft., and the result will be an approximate value for the mean velocity of approach in feet per second.

4. Compute h_v, the head due to this velocity $\left(\text{by the formula } h_v = \frac{v^2}{2g}\right)$, and substitute in the equation above. This will give an approximation to the true discharge.

5. If necessary, repeat the process by recalculating v from this new discharge, and so finding a more exact value of h_v.

As the coefficients proper to be used, in cases where the velocity of approach is great, are uncertain, it is desirable to arrange matters so that this velocity may be, as far as possible, a negligible quantity by raising the weir or excavating a pond.

Bazin—('Expériences nouvelles sur l'écoulement en déversoir,' Paris, 1891)—has, however, determined with great accuracy the effect of velocity of approach in one particular case, namely, the case of a vertical notch extending entirely across a rectangular channel with horizontal bottom, precautions being taken to admit air freely under the apron.

He assumes a general formula, somewhat different from that adopted by English writers, namely,

$$Q = \mu\, l\left(h + a\frac{v^2}{2g}\right)\sqrt{2g\left(h + a\frac{v^2}{2g}\right)}$$

where a is a coefficient whose value is not yet well known, but which is usually taken as $1 \cdot 5$.

This reduces to

$$Q = \mu l h \sqrt{2 g h}\left(1 + a \frac{v^2}{2 g h}\right)^{3/2}$$

where μ is the coefficient of an ordinary notch, where there is no appreciable velocity of approach.

If we write the equation $Q = m l h \sqrt{2 g h}$, to correspond with the formula on p. 225, but using a coefficient m, which includes the velocity of approach, it is evident that m here will be greater than μ, its value being given by the equation

$$m = \mu\left(1 + a \frac{v^2}{2 g h}\right)^{3/2}$$

Expanding this, and neglecting higher powers than the first of the small fraction $\frac{v^2}{2 g h}$, we obtain

$$m = \mu\left(1 + \frac{3}{2} a \frac{v^2}{2 g h}\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, in the particular case experimented upon, namely, where the notch-board extends from side to side of a rectangular channel, if p = height of crest of weir above bottom of channel, it is obvious that

$$v = \frac{Q}{l(p + h)}$$

$l(p + h)$ being the area of section.

Then, putting

$$Q = m l h \sqrt{2 g h}$$

we obtain

$$v^2 = \frac{m^2 h^2 2 g h}{(p + h)^2}, \text{ or } \frac{v^2}{2 g h} = \frac{m^2 h^2}{(p + h)^2}$$

Substituting in equation (1), we obtain

$$m = \mu \left\{1 + \frac{3}{2} a m^2 \frac{h^2}{(h + p)^2}\right\}$$

which we may write

$$m = \mu \left\{1 + K\left(\frac{h}{h + p}\right)^2\right\}$$

where K is a coefficient, determinable only by numerous experiments.

Bazin finally arrived at the result

$$Q = \mu \left\{1 + \cdot 55\left(\frac{h}{p + h}\right)^2\right\} l h \sqrt{2 g h}$$

which does not require any alteration to adapt it to English measures.

The value of the coefficient μ is given on p. 225, and may be taken as about ·42, on the average.

It must be clearly understood that this formula only applies to the special conditions under which the experiments were made, namely, with a rectangular channel having a horizontal bottom and the notch-board extending from side to side, without end contractions.

APPENDIX K.

TIDES IN THE RIVER THAMES.

STATION.	Distance in Miles.	Interval between Stations.	Time of H.W. after Southend Pier.	Time Interval (minutes).	Speed of WaveCrest in Miles per Hour.	Spring Tides.			Neap Tides.			Average Level High and Low Water.	
						H.W.	L.W.	Range Difference.	H.W.	L.W.	Range Difference.	Springs.	Neaps.
Southend Pier	0·0	..	..	..	..	20·67	2·75	17·92	16·67	6·75	9·92	11·71	11·71
Gravesend Pier	17·06	17·06	25	25	40·94	21·25	1·50	19·75	17·08	5·50	11·58	11·37	11·29
Woolwich Ferry	33·94	16·88	65	40	25·32	22·83	1·17	21·66	18·33	5·33	13·00	12·00	11·83
Cherry Garden Pier	42·30	8·36	80	15	33·44	23·42	1·50	21·92	18·75	5·50	13·25	12·46	12·12
London Bridge	43·50	1·20	86	6	12·00	23·50	1·83	21·67	18·75	5·75	13·00	12·66	12·25
Lambeth Pier	45·76	2·26	96	10	13·56	23·58	3·42	20·16	18·83	6·33	12·50	13·50	12·58
Chelsea Pier	48·10	2·34	106	10	14·04	23·58	4·92	18·66	18·83	6·92	11·91	14·25	12·87
Fulham Pier	50·65	2·55	117	11	13·91	23·58	6·08	17·50	18·92	7·50	11·42	14·83	13·21
Hammersmith Bridge	52·61	1·96	125	8	14·70	23·67	7·00	16·67	19·00	7·92	11·08	15·33	13·46
Kew Railway Bridge	56·06	3·45	141	16	12·94	23·75	8·25	15·50	19·25	8·75	10·50	16·00	14·00
Richmond Lock	58·93	2·87	157	16	10·76	24·08	11·50	12·58	19·75	11·50	8·25	17·79	15·62

The levels are referred to a Datum 22·5 ft. below Trinity High-water Mark, or 10 ft. below Ordnance Datum. *Vide* Figs. 142 and 143.

The tides do not afford complete cycles. The method adopted by the Thames Conservancy has been to tabulate the tides for a whole year, omitting those of extraordinary character; 5/28 of the remainder, equal to five tides per semi-lunation, have been taken from the highest tides to represent high-water springs, and 5/28 of the low tides to represent high-water of neap tides, and correspondingly with the low waters.

Below Chelsea the effect of land water has not been taken into account.

This section, compiled from data supplied to the writers by the Thames Conservancy Board, shows clearly the following points:

(*a*) That the velocity of the tide-wave diminishes as it ascends the river, and its section becomes reduced (column v).

(*b*) That the range of the tide becomes greater, up to a certain point, as the wave ascends, and then diminishes again (columns viii and xi). Thus, at Southend the ranges are 17′ 11″ and 9′ 11″ at springs and neaps respectively; at London Bridge they are 21′ 8″ and 13′; whilst at Richmond Lock they are but 12′ 7″ and 8′ 3″.

(*c*) That low water is lower and high water higher at a certain distance up an estuary than at its mouth. Thus, at springs the levels of *low water* are 5′ 3″ at Southend Pier and 4′ 4″ at London Bridge, whilst at Richmond low water is, at *springs* and *neaps*, 14′.

(*d*) Similarly, high water is higher at London Bridge than at Southend Pier.

(*e*) That the principal difference in range is due to the rise of the low-water mark. Thus, high-water mark springs at Richmond is 26′ 7″ − 23′ 2″ = 3′ 5″ above Southend, whilst the difference at neaps is 22′ 3″ − 19′ 2″ = 3′ 1″.

From columns vi and ix it appears that the level of high water is an approximately uniform, but slightly increasing, quantity. The level of low water, however, falls to about London Bridge and then rapidly rises again to Richmond. (Fig. 143.)

(*f*) The differences of range are therefore due rather to differences in the level of low-water mark than to differences in the level of high water.

(*g*) This points to the fact that the mean river level rises above the mean sea level as the river is ascended.

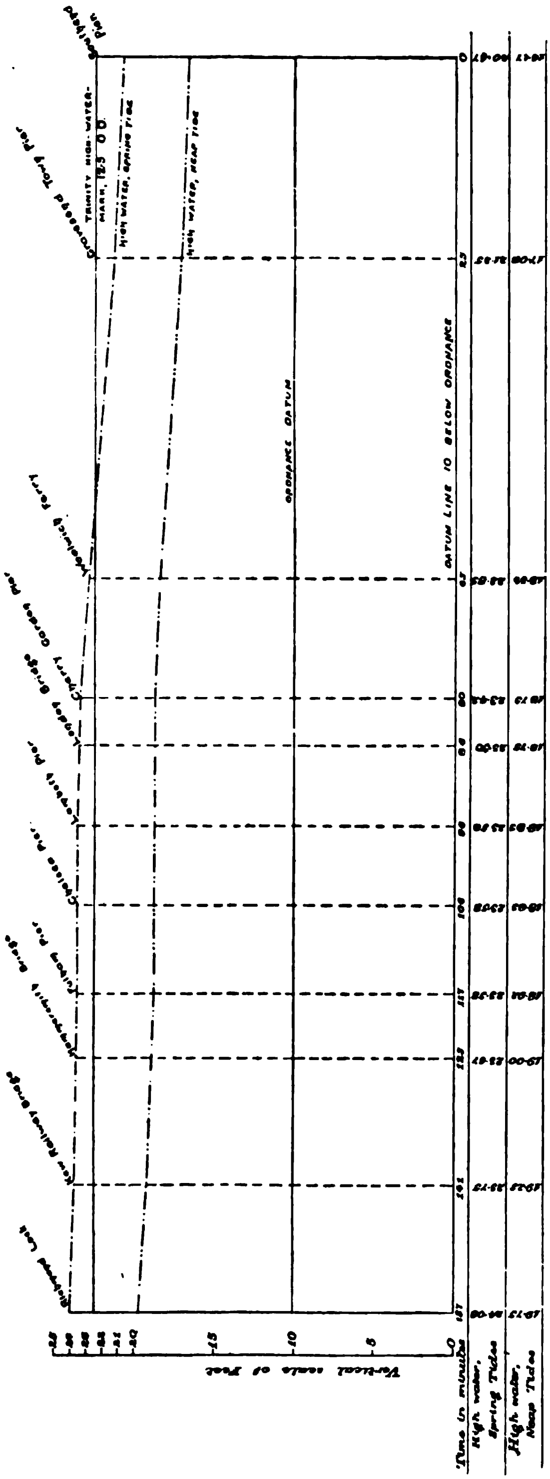

TRINITY HIGH-WATER-MARK, 12.5 O.D.
HIGH WATER, SPRING TIDE
HIGH WATER, NEAP TIDE
ORDNANCE DATUM
DATUM LINE 10 BELOW ORDNANCE
Vertical scale of Feet
Time in minutes
High water, Spring Tides
High water, Neap Tides

FIG. 142.

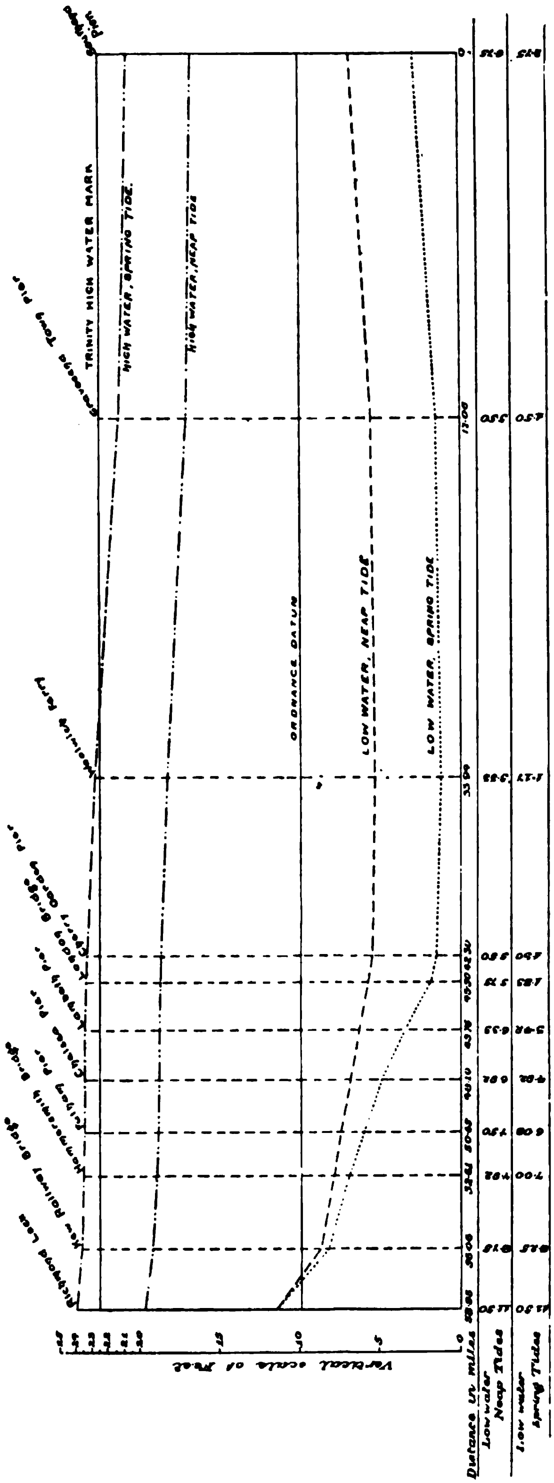

TRINITY HIGH WATER MARK
HIGH WATER, SPRING TIDE
HIGH WATER, NEAP TIDE
ORDNANCE DATUM
LOW WATER, NEAP TIDE
LOW WATER, SPRING TIDE
Vertical scale of Feet
Distance in miles
Low water Neap Tides
Low water Spring Tides

FIG. 143.

APPENDIX L.

Engineering Plans of Harbours, Estuaries, etc.

The survey of the area covered by high water of ordinary spring tides is made by traversing in the usual manner.

In an open harbour the level of high water of ordinary spring tides may be fixed by observation, and a contour levelled in with the levelling staff, the points of change of direction to any required degree of nicety being surveyed with the theodolite, chain and offsets, or the whole operation, including the levelling, being carried out tacheometrically.

In long and intricate estuaries the level of high water of ordinary spring tides cannot be taken as invariable, and observations for ascertaining the true level should be made at distances of a mile or so. When these have been ascertained the survey may be proceeded with by traversing, allowance being made in the levelling for the greater height to which the tide rises as it recedes from the open sea.

A survey of the area covered by water at low water of ordinary spring tides may be arrived at in the same manner, but as the land is uncovered, at the longest, for an hour or two out of the twenty-four, and during many days in any month is covered with water, the process is tedious, and the desired result may be arrived at more rapidly by including this part of the survey in that to which this article particularly refers, namely, in that which is submarine.

In land-locked harbours, and in lakes where there is little or no current, the simplest method to pursue is to set out parallel lines from well-defined leading marks on the shore, fixed presumably when the survey for high-water mark was made, or which can be otherwise identified. The boat from which the sounding operations for ascertaining the depth of the bottom beneath the surface of the water are conducted should follow along these lines, the distance of the boat from given points on the shore being determined by angles taken from the boat with a sextant between the line of direction already referred to and well defined fixed shore marks, such as church spires, or clock tower, etc. The angle should be as large, up to 90°, as possible.

Lines having been drawn and soundings taken in one direction across the harbour or lake, similar lines should be drawn and soundings taken on them in a direction nearly perpendicular to the first.

The number of soundings which it is necessary to record must depend on the formation of the ground. If the slopes are long and regular in their inclination few observations are required, but a sounding must be recorded at each change of gradient, and as the position of changes cannot be observed by the eye, the

sounding line must therefore be constantly going, though it may not be necessary to record all the observations.

As the readings on the sounding line measure the distance from the surface of the water to the ground, and the water level is constantly fluctuating, an observer must be told off to record the level of the water measured on a gauge referred to high-water mark or some other datum at, say, every five minutes, and the time at which each sounding is made must be booked.

The observations should be taken from still, or nearly still, water, which may be obtained by sinking a vertical cast-iron pipe of large diameter, say from 24 in. to 36 in., in the deepest available part of the harbour, attaching a cross pipe of, say, 9 in. diameter to the lower end of the vertical pipe, suspending a float of only slightly smaller diameter than the main pipe in it, and recording the levels by means of a scale and pointer attached to a wire fixed to the float.

In large and important surveys it will be more satisfactory and more economical if an automatic tide-recorder be attached to the float, and the time and recorded level of the water be read off the diagram card. The attendance of an observer may then be dispensed with.

If great accuracy be required, the position of the sounding line at the moment of taking a level may be observed by two theodolites, one placed on the line of observation, the other as nearly as is practicable in a position at right angles to the centre of the line of observation, supposing it to be practicable to observe every station from one setting up of the theodolite.

In estuaries and harbours where there is a considerable velocity of current, the system of observation described above is inapplicable; there is slack water for only some thirty minutes at high and low tide, and if the boat from which the soundings are taken were held steady during the taking of the measurement, the sounding line would be so materially deflected from the vertical by the action of the tide as to render the observation invalid.

Under the above conditions it is desirable that the boat should float with the tide, and should be in motion at all times except at slack water, even during the time of sounding.

The actual sounding must occupy the least time possible, so that the line may not be carried out of the vertical by the tide or by the drag of the boat.

The position of the sounding-line will be recorded by observations from two, or preferably three, theodolites, placed on the shore in known positions, the observer in the boat signalling to those on shore at the exact moment of making the observation.

The survey may conveniently be made in lengths of from a quarter to half a mile at a time, so that the whole distance may be within view of the two or three theodolites without shifting.

The boat may be started to drift along the side of the estuary nearest to the observers, and watermen will easily so handle their boat, without materially increasing or retarding its way, that it shall travel along nearly parallel lines, starting a little further from the bank for each succeeding set of observations.

When the work has been plotted, should any part of the chart be unoccupied, special observations should be taken for filling in the vacancies.

In estuaries it is requisite that the observations for determining the level of the water at each five minutes should be taken within half a mile of the position of the soundings to which they refer, the surface of the water not being horizontal but on an inclined plane. The system of using an automatic recorder is not therefore applicable unless one be placed at every mile, or unless sufficient observations have been made to determine what will be the height of the tide at any part of the estuary relative to a given position on the automatic recorder placed in a fixed station.

A full system of signalling should be arranged between the observers in the boat, of whom it is preferable to have two, so that any sudden change of level may not pass unnoticed, and who will check one another and those on shore, as it may frequently happen that one or other party may wish to have an observation repeated.

Careful records of the time when each observation was made, and of its number, should be made by both parties.

The provision of a satisfactory sounding-line is a somewhat difficult problem. A rope, the line which is most easily manipulated, is quite unreliable; a stranded brass or copper rope stretches and becomes inaccurate after short usage; a steel sounding-line is difficult to handle, and the attachment of tags for recording the depth at each foot is unsatisfactory, as the wire, being hard, cannot be coiled by hand but must be laid on to a drum, and the edge of the drum meeting the tags pulls them off. A steel-cored coiled brass wire rope would answer all the requirements, as it can be handled and coiled with fair facility, the tags can be attached to the steel core, and it is not necessary to use a drum, were it not that immersion in salt water sets up galvanic action between the brass and the steel, and the latter is very soon destroyed.

Probably the most suitable rope is to be found in a steel wire, the feet marks being made with soft iron wire coiled and soldered on, and with an iron or steel link attached to each, to which the corresponding numbered tag is fixed by means of iron wire and also with rope, the whole of the steel line being covered with plaited hemp rope soaked in petroleum, to prevent, so far as is practicable, the rusting of the steel core and its attachments.

APPENDIX M.

TACHEOMETRIC FIELD-BOOK.

THE tacheometric field-book in use at University College, London, is ruled into fifteen columns as follows:—

(1) Station and zero; (2) Point; (3) Bearing or horizontal angle; (4) Readings; (5) Generating number (cs); (6) Zenith distance; (7) Angle of elevation (d); (8) $cs \sin \alpha \cdot \cos \alpha$; (9) Theodolite, and middle reading; (10) Difference; (11) Rise; (12) Fall; (13) Reduced level; (14) Horizontal distance; (15) Remarks.

The printed rules for observing. and for filling-in the book, explain that on setting up the tacheometer at any station, the name (or number) of station is filled in in column (1). If the telescope be directed to magnetic north as zero, this is also filled in in the first column, thus Mag. Nth = 0. The height of the theodolite is measured and filled in in the first line of column (9). The reduced level of the station is filled in, when known, in column (13). The rest of the columns are left blank in the first line.

It is then advisable to take a complete set of back readings to the back station. This tends to lessen any errors due to curvature or refraction. In this case the name (or number) of the back station is entered in column (2), line 2. If magnetic bearings are being read, the back bearing will appear next in column (3), and a comparison with the forward bearing will show any error in the compass. But if included angles are being read, the theodolite would be set to zero on the back station, and 0° 0′ would be filled in in column (3), line 2, to indicate this, the reference "Mag. Nth. = 0" in column (1) being, of course omitted.

In column (4) the readings of the top and bottom hairs are entered, and bracketed. The difference between these, multiplied by the constant of the instrument, gives the "Generating number," column (5).

Column (6), Zenith distance, gives the angular distance from the zenith. Most English tacheometers give this angle on the vertical limb, instead of the angle from the horizontal.

This angle, as observed, will be less than 90° for elevation, and more for depression. It is entered in column (6).

Column (7), Angle of elevation, is got by *subtracting the Zenith distance from* 90°.

Thus it will be plus for elevation, and minus for depression, and must be entered plus or minus accordingly.

With instruments graduated from the horizontal both ways, column (6) may be omitted, if the tables are computed from similar angles, but care must be taken to see that the proper sign is entered for each vertical angle.

Column (7) for angle of elevation is added, because, at University College,

some of the tables used (Jordan's) are computed from the angle with the horizontal. But if such tables as Kennedy's (computed from the Zenith angle) are exclusively used, this column may be omitted if desired.

Column (8) is obtained from tables. It will be plus or minus according to the sign of column (7).

The reading of the *middle hair* is entered in column (9), line 2. This *subtracted from the height of the theodolite* will give the value of the "difference" to be entered in column (10). The height of the theodolite is, of course, in line 1, column (9), and will remain the same until the station is changed.

The difference (column 10) will be plus, if the height of the theodolite is greater than the middle reading, but minus if less. It must be entered plus or minus accordingly.

The results of column (8) and column (10) are next combined, with due attention to the signs. If plus, the result is a rise, and if minus a fall, *from* the tacheometer station *to* the point observed.

The reduced levels are worked out as in ordinary levelling. Column (14) gives the result of $c\,s\cos^2\alpha$, and is obtained from tables.

It is a good plan, if great accuracy is aimed at, to take all readings between stations both face left and face right. This will mean two lines for the back readings, and the face is entered in the remarks column.

The field-book is bound with interleaved blank pages for sketches.

The back readings complete, other points are, of course, observed, corresponding numbers for them being shown in the Point column and on the sketch.

All computation of co-ordinates, etc., is done on separate sheets.

APPENDIX N.

The Greek Alphabet of 24 Letters.

Α	α	Alpha	Ν	ν	Nu
Β	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	Ο	ο	Omicron
Δ	δ	Delta	Π	π	Pi
Ε	ε	Epsilon	Ρ	ρ	Rho
Ζ	ζ	Zēta	Σ	σ	Sigma
Η	η	Ēta	Τ	τ	Tau
Θ	θ	Thēta	Υ	υ	Upsilon
Ι	ι	Iota	Φ	φ	Phi
Κ	κ	Kappa	Χ	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
Μ	μ	Mu	Ω	ω	Omega

APPENDIX O.

A Table giving the English Equivalents of Metric Measures.

Lineal Measures.

1. *Lineal Measures.*

A centimetre = 0·3937 of an inch
A metre = 39·37 inches = 3·281 feet or 1·094* yards
A kilometre = 0·6214 of a mile

Square Measures.

2. *Square Measures.*

A square centimetre = 0·155 of a sq. inch
A square metre = 10·764 sq. feet or 1·196 sq. yards
A hectare = 10,000 sq. metres
= 2·471 acres

Cubic Measures.

3. *Cubic Measures.*

A cubic centimetre = 0·06103 of a cub. inch
" metre = 35·32 cub. feet = 1·308 cub. yards
A litre = 1000 cub. centimetres
= 61·03 cub. inches = 1·761 pints
= ·02201 of a gallon

Mass.

4. *Mass.*

A gramme = 15·43 grains
= 0·03527 of an ounce (avoirdupois)
= 0·03215 of an ounce (troy)
A kilogram = 2·2046 pounds avoirdupois
A metric ton = 1000 kgs. = 0·9842 of a ton of 20 cwts.

* A close approximation to the length of a metre in feet is, 1 metre = 3 feet $3\frac{3}{8}$ inches. The metre was originally intended to be one ten-millionth of the length of an arc of the meridian, from the pole to the equator, but owing to errors in determining, this length is now legally defined as $\frac{443296}{864000}$ of the length of a bar called the Toise of Peru. Strictly, therefore, the metre has no greater scientific accuracy than the British yard, which is the distance between two marks on a bar kept in the palace of Westminster.

APPENDIX P.

Notes on Photography from Balloons and Kites as Applicable to the Art of Surveying.

1. Balloon Photography.

Balloon Photography. A map of an area passed over by a balloon carrying a camera (whether with an observer, or sent on a free flight, or captive) can be made on the basis of determining (by recognition) several points in the landscape identifiable on the photographic print. If one photo alone be taken, the survey which can be evolved therefrom will be on the assumption that the land is plane (without undulations or hills, etc., as found in nature), whilst if several photos be taken from several points (only possible of course with an observer aloft) the vertical features can be determined. The "modus operandi" for working out these features are beyond the scope of this Appendix, but are fully discussed in a "Pocket-Book of Aeronautics," by Major Hermann W. L. Moedebeck, of which a translation is published by Whittaker and Co., 8 White Hart Street, Paternoster Square, E.C.

2. Kite Photography.

History of. Kites capable of raising the necessary instruments were first used for scientific purposes about 150 years ago, e.g. by A. Wilson in 1749 and Franklin in 1792.

Their use for meteorologic purposes was first realised scientifically in the United States in the year 1883, and in connection with these uses the first apparatus with clockwork was sent up by means of a kite from Blue Hill in the year 1894.

Types of Kites. Kites have been made of every conceivable form and arrangement of lifting planes, but the two most suitable for the purposes under consideration are the box pattern and the ordinary kite (of our youth) or Malay kite. For reasons of facility and simplicity of construction and in repairs, the writer gives the preference to the latter form of kite. It is most easily handled when not exceeding 6 feet 6 inches to 7 feet in length, and if the lifting power be not sufficient a series of kites can be used in tandem. When so used the distances apart must not exceed 20 to 30 yards, or varying currents of air may interfere.

Experiments in France in 1880.

M. Batist sent up a kite in 1880 of the Eddy type (of Malay pattern). It was 8 feet 3 inches long and 5 feet 9 inches wide and weighed 4 lb., the camera and cord, etc., being of the same weight. The exposure was made by means of a time fuse. The camera was fixed to the backbone of the kite, and the yokes bifurcated by use of a spreader (of say 2 feet in length) to clear the downward line of sight.

It may sometimes be more convenient to send up the camera to the kite when it is aloft, and this can be done in several ways: (*a*) by means of a collapsible carrier kite (with a light restraining string to prevent its taking charge in a high wind). The writer has found this method very practical. The camera is carried in gimbals fixed to a long (say 3 foot) light wooden carrier on 1 inch wheels, and can be set to any required vertical or horizontal angle. On reaching a *striking plate* of cardboard, near the kite, a wire is pressed back and releases a small weight, say ½ oz. to fall some 9 inches and release the shutter, and a weight of 1 lb. or so, with a drop of say 5 or 6 feet, to draw a pin and release the restraining strings of the carrier, which then collapses, and sails down into the hands of the observer; (*b*) by means of a double string running through a pulley near the kite, just above which a striking plate of cardboard is fixed (as above described). The carrier and striker, with camera fixed as above, is interpolated in the loose end of the string when the kite has reached *half* the desired height and is then eased up by hand. The exposure is made in the same way as with a carrier kite.

The tail used with the Eddy or Malay kite may be made either with paper or linen strips, or by a more modern invention, being frustra of cones made of light material on wire circles (say 5 or 6 inches and 3 or 4 inches in diameter), strung on to the string. With these a much shorter tail will have the desired effect, say 5 or 6 frustra set at 18 inches to 2 feet apart.

The box kite, of course, requires no tail, but in the opinion of the writer has many disadvantages as compared with the Malay type, being more costly and difficult to repair.

The String.

For reaching great altitudes steel wire is the only material suitable, though for short lengths good hemp or China grass strings may be found desirable and more easy to handle.

The following table gives an idea of the breaking strains set forth in it :—

Breaking Stress in kilograms (2·2 *lbs.*).

Diameters	50	100	150	200
Steel wire . .	0·48	0·67	0·86	1·06
Hemp cord . .	2·40	3·30	3·90	4·60

The diameters are in millimetres.

Allow a factor of safety of one-half of the above, for safe breaking strain.

Keep wire slightly greased and dry.

Tinned wire has some advantages.

APPENDIX Q.

Deacon's Cross-sight Attachment to the Surveying Level.
(Fig. 145.)

This instrument, which is manufactured by T. Cooke & Sons, is designed to enable the surveyor to set out cross-sections at right angles to a main line without the use of any instrument other than the level.

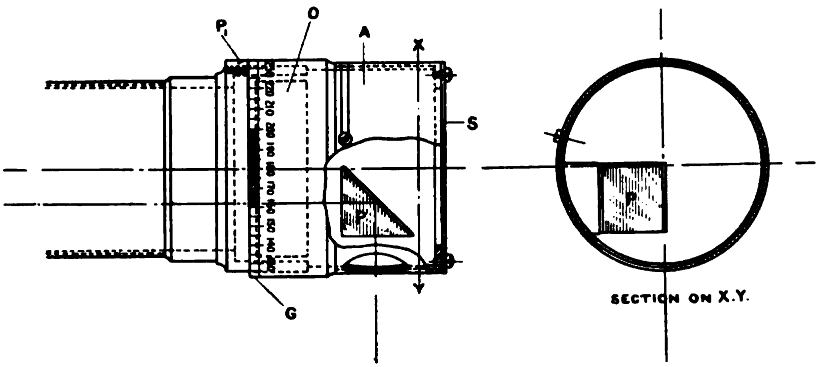


Fig. 145.—Deacon's Cross-sight Attachment.

The instrument consists of a small right-angled prism placed in the sun-cap of the level. The prism is placed so that the reflecting face makes an angle of 45° with the line of collimation of the telescope, and opposite to the reflector a small hole is cut in the side of the cap; this hole can be closed by means of a slide.

If the slide be closed and the end of the cap opened the level is used in the ordinary way, the prism being invisible. On closing the end of the cap and opening the slide opposite the prism the line of sight is bent through a right angle, and on looking through the telescope objects at right angles to the axis of the telescope will be observed. The cap is capable of being turned on the telescope so that lines may be ranged up or down hill; the cap also has a scale of degrees engraved round its edge, so that angles of elevation or depression may be observed.

The prism occupies one-fifth the area of the object glass.

It is claimed that this instrument replaces the optical square for setting out cross-sections and in all cases where it is necessary to set out right angles before levelling.

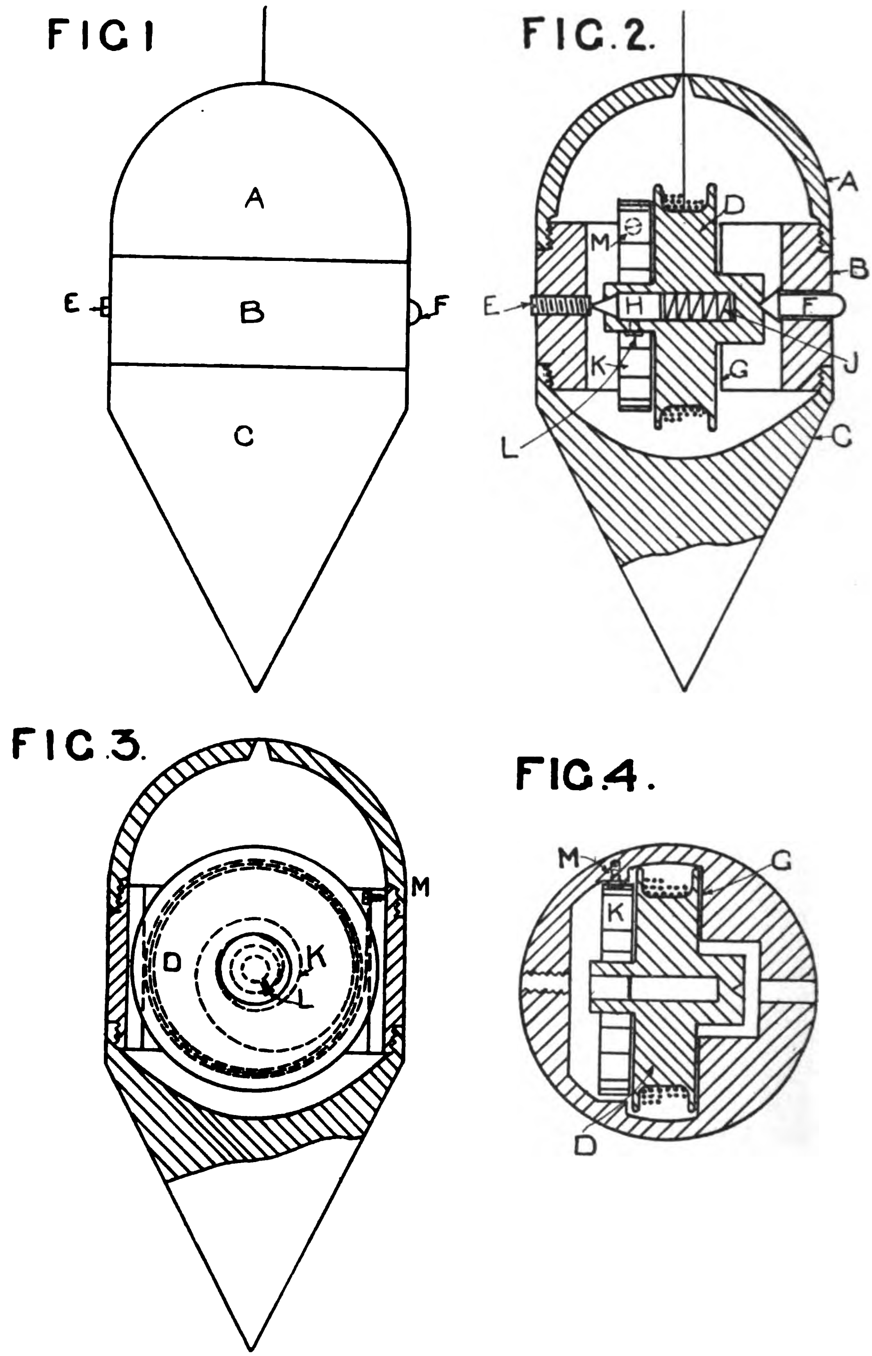


FIG. 146.

DEACON'S ADJUSTABLE PLUMB-BOB. (Fig. 146.)

The plumb-bob is cast in three parts, A, B and C, which are capable of being screwed together as in figs. 1, 2 and 3. The castings are hollowed out as shown in section in figs. 2 and 3, and in plan, fig. 4, and the wheel D, figs. 2, 3 and 4, is carried by the bearings E and F, figs. 1 and 2. This wheel is held stationary by the friction between one of its sides and the side of the casting at G, figs. 2 and 4. The small metal cylinder, H, fig. 2, fits into part of the space I (fig. 4) and rests against the spring J (fig. 2), which fills the remainder of the space. The spiral spring, K, figs. 2, 3 and 4 is attached to the hub of the wheel at L, figs. 2 and 3, and to the casting at M, figs.2, 3 and 4. On pressing the button F, figs. 1 and 2, the wheel is shifted laterally from the face of the casting at G (fig. 2), and relieved of its friction, and the spring J, fig. 2, is compressed. The plumb-bob can then be raised or lowered to the desired position. When the button F is released, spring G throws the wheel into friction again, and the plumb-bob is held stationary. When the plumb-bob is lowered, spring K is wound up. When it is desired to raise the plumb-bob, press the button F; the wheel D will then be released and the spring K will cause it to revolve and to wind up the cord of the plumb-bob.

The plumb bob hangs quite freely without any movement, in any position.

CASELLA'S GRADIENT TELEMETER LEVEL. (Fig. 147.)

This instrument is designed for measuring linear distances, gradients and differences of level by one and the same observation without the use of chain or tape.

It is claimed for the instrument that the measurement of distances is done more accurately than with the chain, and that more work can be executed in a given space of time than can be secured with the surveying instruments in general use.

Description of the Instrument.

The Gradient Telemeter Level is similar in general construction to the Y level, but with the addition of a circle, on which are marked the gradients from 1 in 1200 to 1 in 10, or any other series of gradients determined on. The gradient marks are such a distance apart on the circle that they can be easily read at the index without the aid of a reading microscope or vernier.

The principle of the instrument is that the telescope is horizontal only when the index is at zero; in other positions the telescope points upwards or downwards to such amount as is indicated on the circular inclined plate on which the several gradient lines are engraved.

Construction of the Instrument.

The construction of the instrument differs from that of the Y level in that the support for the Y's which carry the telescope is inclined and that there are two axes. The lower axis which connects the trybrach with the divided circle is vertical, while the upper axis, which connects the circle with the Y supports of the telescope, is inclined and perpendicular to the plane of the circle.

Use of Instrument. The instrument may be used for taking approximate gradients. Set up and read the height of the telescope by means of the tape plumb-bob provided, then sight the staff by directing the telescope towards the staff and keeping it in this position, holding it lightly with the fingers, then rotate the circle until the cross hair cuts the staff at exactly the same height as is shown on the tape, then the gradient may be read off the circle; as the circle is only coarsely divided, it is not possible to read the gradient with greater accuracy than can be obtained by estimating the parts into which the index divides a division of the circle.

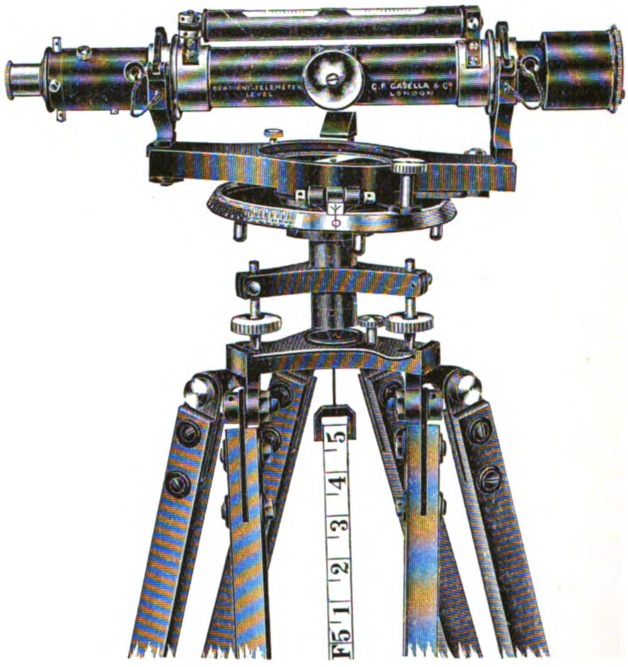


FIG. 147

Distances may be obtained as follows. Sight the staff near the bottom, set the index exactly to the nearest convenient division on the circle, say 1 in 50, and read circle and staff; now set the index to, say, 1 in 100 and read again. The difference of the staff readings multiplied by 100 will give the true horizontal linear distance between the instrument and the staff. It is advisable that the gradients selected should be from one of the gradient pairs marked on the undivided side of the circle, as otherwise the multiplier will not be 100 and the calculation necessary to obtain the distance will be more complicated.

The Gradient Pairs are as follows:—

$$\frac{\text{level}}{100}\Big\}, \quad \begin{matrix}100\\50\end{matrix}\Big\} \quad \begin{matrix}66\frac{2}{3}\\40\end{matrix}\Big\} \quad \begin{matrix}60\\37\frac{1}{2}\end{matrix}\Big\} \quad \begin{matrix}50\\33\frac{1}{3}\end{matrix}\Big\} \quad \begin{matrix}33\frac{1}{3}\\25\end{matrix}\Big\} \quad \begin{matrix}25\\20\end{matrix}\Big\} \quad \begin{matrix}20\\16\frac{2}{3}\end{matrix}\Big\} \quad \begin{matrix}12\frac{1}{2}\\11\frac{1}{9}\end{matrix}\Big\} \quad \begin{matrix}11\frac{1}{9}\\10\end{matrix}\Big\}.$$

If one of the gradients selected is double the other the higher gradient may be used as the multiplier, thus $\left.\begin{matrix}60\\30\end{matrix}\right\}$, multiplier 60.

To obtain differences of level set the index at some convenient division of the circle and read the staff; then the distance obtained, as above, divided by the gradient, gives the difference in level between the telescope axis and the point sighted on the staff, and to find the difference of level at the ground it is only necessary, in the case of a fall, to add the reading of the staff and deduct the height of the instrument, and in the case of a rise, deduct the staff reading and add the height of the instrument above the ground. To read rises or falls the telescope is reversed end for end in its Y's; one end of the Y frame is marked "rise" and the other end "fall," and the eye-piece end of the telescope indicates whether a rise or fall is being observed.

Example.—Suppose the height of the telescope above the ground is 4·50 feet, sight 4·50 on the staff and the index will show the gradient on the circle. As before stated this reading can only be taken approximately. Now set the index to the nearest gradient line, say 1 in 50, and read the staff, say 4.55; again set the index to the gradient 1 in 100 and read the staff, say 12·55, difference 8 feet; then the distance of the staff from the instrument is 8 × 100 = 800 feet and the difference in level is,

$$\text{if a fall } \frac{800}{50} + 4{\cdot}55 - 4{\cdot}50 = 16{\cdot}05 \text{ feet.}$$

$$\text{if a rise } \frac{800}{50} - 4{\cdot}55 + 4{\cdot}50 = 15{\cdot}95 \text{ feet.}$$

The staff must be observed between the vertical lines on the diaphragm.

Ferguson-Steward's Pedograph: An Automatic Road Tracer. (Fig. 148.)

The pedograph is designed with the object of producing a rapid and fairly accurate traverse by mechanical and nearly automatic processes.

The instrument consists of two plane parallel surfaces, mounted so that they can be revolved in a flat case, which is carried at the side of the operator by a shoulder strap. On the top of the case is a compass, above which is an indicator geared to the plane surfaces, revolving with them through a corresponding angle, so that the operator can see at a glance the position of the plane surfaces. A piece of drawing paper is attached to one of the surfaces, and receives the tracing made by the automatic recorder

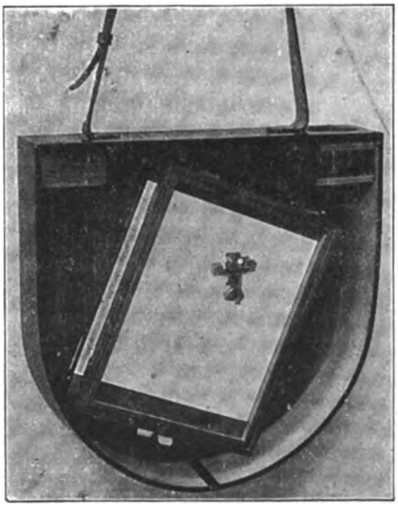


Fig. 148.

which is held suspended between the two planes. The action of walking causes the recorder to travel across the paper, and in its course it traces a facsimile of the route walked over.

All the operator has to do is to walk over the road he wishes to map out with the "pedograph" suspended from his shoulder, and by means of a knop projecting from the side of case preserve coincidence between the compass needle and the indicator.

Reeves' Proportional Dividers. (Fig. 149.)

The instrument consists of an ordinary pair of dividers, fitted with a movable scale, by means of which proportional measurements can be made, lines divided into any number of parts, and latitudes and longitudes accurately read from

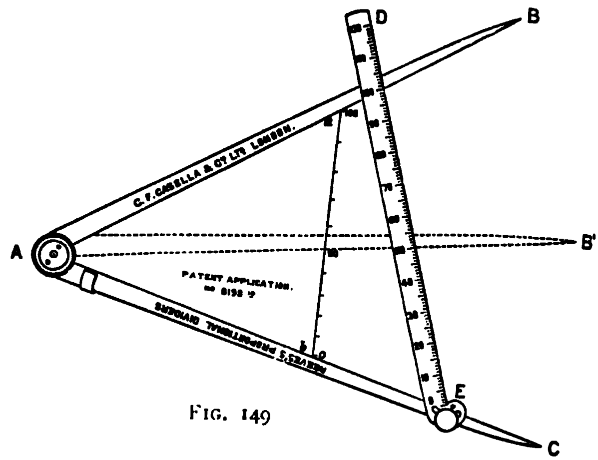


FIG. 149

maps and charts. It will serve as a diagonal scale or vernier for exact measurements of various kinds, etc.

The Steward Hypsometric Aneroid. (Fig. 150.)

It is claimed for this instrument that by its use the measurement of altitudes is simplified.

Description. The barometric portion of the instrument does not differ from those in general use.

The altitude scale forms a complete circle of equal divisions, and the zero is adjustable. Ascents and descents are placed to the left and right of zero respectively. The altitude scale is automatically locked and cannot shift during transit.

It is claimed for this instrument that the difference of level between stations can be read directly without calculation. With an altitude scale extending to 10,000 feet, differences of 5 feet or less can be read without the use of a vernier, and there is no error of parallax.

A small swing thermometer, for ascertaining the air temperature, to make the correction for difference in the weight and volume of the atmosphere is attached.

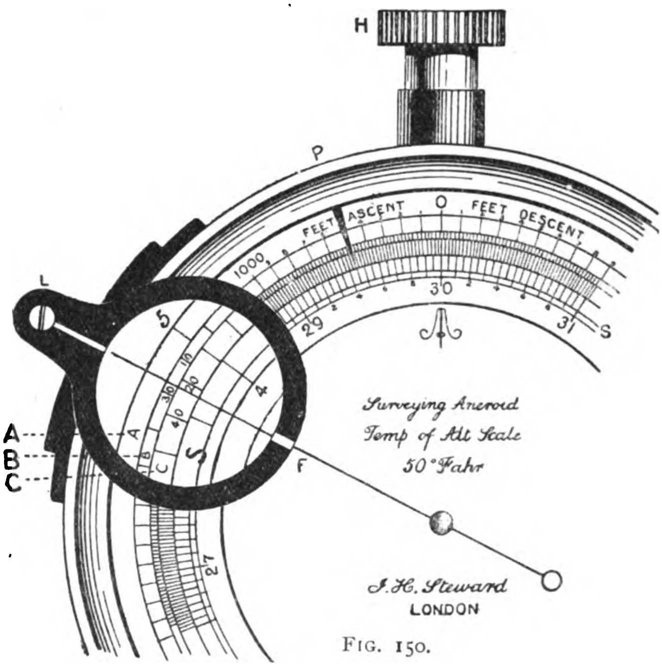


FIG. 150.

Method of Reading.

The lens L, which is attached to a movable ring, must be drawn out about 1 inch, so that its rod can pass over the pinion H. Set the lens over the index hand F. Adjust the lens arc in the same line. In the figure the index hand F is to the left of zero, consequently it indicates an "ascent." Reading from zero to the left the index hand is seen to have passed 1500 feet.

THE STEWARD SURVEYING TELEMETER. (Fig. 151.)

It is claimed for this instrument that by its use distances may be measured with considerable accuracy, and that surveys may be made with it without any other optical instrument being employed. The makers quote nineteen observations where the average length per observation was 1728 feet, the longest measured 2786 feet, and the shortest 649 feet. The average plus error was 16·6 feet, the average minus error 17·4 feet, and the average error was 17·1 feet, or 1·01 per cent. nearly.

Description of Instrument.

The instrument consists of a cylindrical case containing two mirrors, both of which are adjustable. The instrument is designed to measure the two angles at the extremities of a base opposed to the object observed.

By rotating a collar at the end of the tube the index mirror is moved in

azimuth, and the angular displacement is measured by reference to the distances graduated on the exterior of the collar.

The angle of double reflection can be varied several degrees on either side of the right angle, facilitating the determination of the direction of the base.

The range is read directly on the graduated scale engraved on the collar, in terms of units of the base.

FIG. 151.

The base, if short, may be measured with a tape, and should be approximately at right angles to the observation line.

Longer bases may be measured with the instrument itself from an original tape measurement.

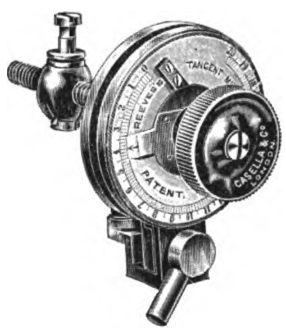


FIG. 152.

REEVES' TANGENT MICROMETER. (Fig. 152.)

This instrument, which is manufactured by C. F. Casella and Co., is a micrometer attachment working on the tangent slow motion screw of a theodolite, it can be fitted to any existing theodolite, and owing to its position on the tangent screw, only one is necessary on each circle.

It is claimed that this instrument is simpler and cheaper than the ordinary micrometer, and that both from its position and construction it is less liable to damage.

FERGUSSON'S PERCENTAGE THEODOLITE.

The Percentage Theodolite.

The percentage theodolite is a rapid surveying and telemeter instrument, in which the circles can be read both in degrees, minutes and seconds, and also in percentage units of angular measurement.

It is claimed that:—

(1) It does away with traverse tables, and with tables for the reduction of the rod for inclined sights.

(2) The engineer can always determine where he is and at what level he is, without having to wait until he gets home to work out his courses, distances, and levels.

(3) With the percentage theodolite, the engineer can immediately range out curves of any radii without the use of curve tables.

The method of forming the percentage unit division is to inscribe a circle inside a square and draw the four quadrantal lines, which divide the circle into quadrants, then draw diagonal lines through the four corners of the square, thus dividing the circle into octants. The four lines of the square will form tangents

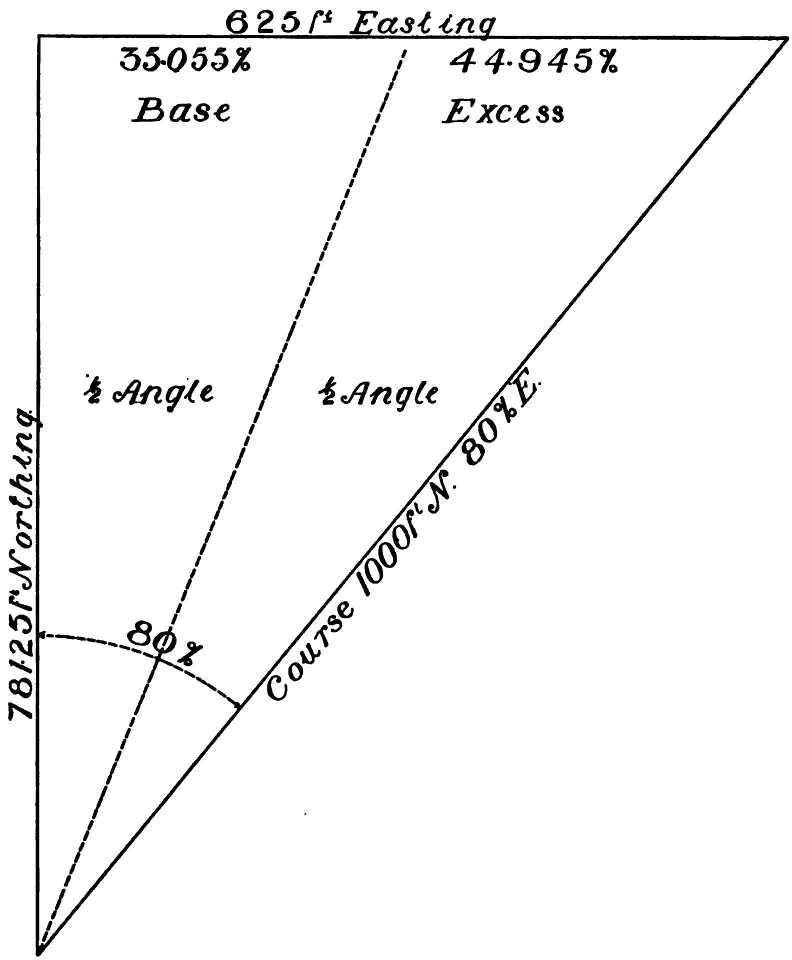


FIG. 153.

to the eight octants. Divide each of the eight tangents into 100 equal divisions, and then divide the octant arcs into 100 spaces by lines, running from the centre of the circle, through the octant arc towards the 100 divisions of the tangent. Every division of the octant will now subtend a space of $\frac{1}{100}$ of the tangent, and also $\frac{1}{100}$ of the radius.

Anyone can immediately tell the departure contained in an angle of 40 per

cent. at 380 feet from the vertex, because 40 per cent. of 380 feet measures 152 feet.

Owing to the percentage division of the octant being of unequal dimensions, that is to say, on account of the divisions being larger near the quadrantal base line; if any angle is bisected, the half nearer the base line will contain fewer percentage units, or divisions, than the half of the same angle which is farther from the base line, thus

Take, as an example, a course of 1000 feet long, bearing north 80 per cent. east; when the angle of 80 per cent. is bisected, the half of the angle adjacent to

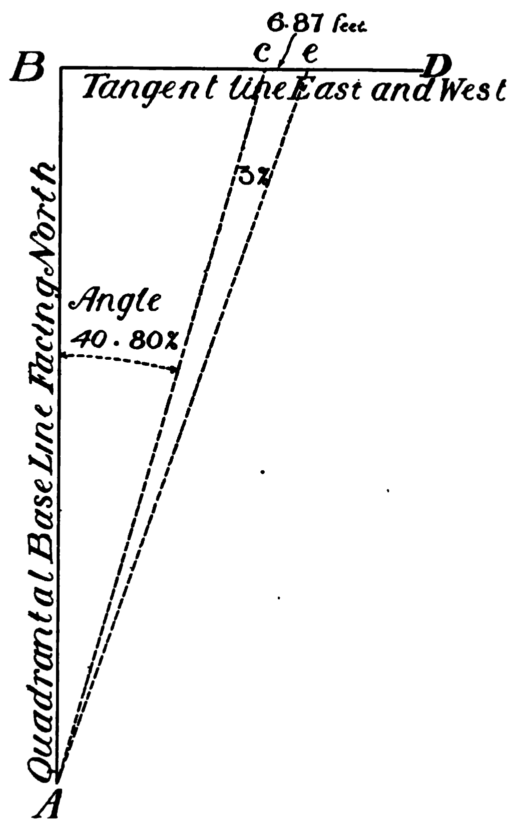


FIG. 154.

the base line contains 35·055 divisions, and the other half of this angle farther from the base line contains 44·945 divisions. These two numbers express the ratio which the length of the 1000 feet course (north 80 per cent. east) bears to the quadrantal base line running due north; so that every 44·945 feet along the course will contain 35·055 feet of Northing along the quadrantal base line (see Fig. 153).

Therefore, the length of the course is 1000 feet, the Northing is 781·25 feet, and the departure or Easting is 80 per cent. of 781·25, or 625 feet. In this way, the latitude and departure of a course measured in any direction is immediately

established, and the tangent, sine, secant, radius, or cosine can be obtained by using these ratios; for in an angle of 80 per cent. :—

$$\begin{array}{ccccccc} 44{\cdot}945 & : & 35{\cdot}055 & :: & \text{Tan.} & : & \text{Sin.} \\ ,, & : & ,, & :: & \text{Sec.} & : & \text{Rad.} \\ ,, & : & ,, & :: & \text{Rad.} & : & \text{Cos.} \end{array}$$

Example in Use of Instrument.—Should the quadrantal lines of the circle be set facing the cardinal points of the compass, as in Fig. 154, and the object C lie north 40·80 per cent. east. Find the latitude and departure of the object C.

Let the space covered, anywhere along the tangent off-set B C by three divisions of the octant, measure 6·87 feet, then $\frac{6{\cdot}87 \times 100}{3} = 299$, the Northing of the course A B.

The Easting of the course A C is the distance B C, 40·80 per cent. of the distance A B, and 40·80 per cent. of 299 feet = 93·63 feet.

But Northing is latitude and Easting is departure, so we have:—

The direction of C. North 40·80 per cent. East.
The latitude of C. 229 feet North.
The departure of C. 93·63 feet East.

This method does away with the necessity of traverse tables for determining the latitude and departure of a course.

Note.—Should a space on the tangent offset be subtended by a decimal portion of a division on the octant arc (thus 2·48 feet by decimal 8 of a division), simply multiply by 1000 and divide by 8 as $\frac{2480}{8} = 310$ feet, the length along the base line.

The Tape Plumb-bob. This is a plumb-bob suspended by a tape from the stirrup screwed under the centre of the instrument. The plumb-bob is adjusted to hang just clear of the peg, over which the instrument has been set up; the reading of the tape at the stirrup will then correspond exactly with the height of the instrument at the axis of the telescope. Sufficient allowance is deducted from the tape for the height of the plumb-bob and the distance between the stirrup and the axis of the telescope.

Water Level Recorder, with extended scale. This apparatus is particularly adapted for use in harbours and for many other purposes, in addition to the recording of the movements of tides.

The clockwork movement is bolted to the iron base of the apparatus, and well protected from the weather. The clock may revolve once in 7 days or in 24 hours.

Water Level Recorder, as described above, but with apparatus for continuous paper. In many cases it is desired that the rate of progress of the paper should be as rapid as possible, and yet it would not be convenient to change the chart every day, and for such cases this Recorder is constructed with an arrangement allowing of the use of a roll of continuous paper which is sufficient to last one month. The time is indicated on the paper by means of

the impression of a series of steel pins, which are shown at the upper end of the drum. The advantage of the continuous paper is that it is only necessary to wind the clock once a week, and there is no risk of the loss of diagram through the prolonged absence of the attendant.

Tide Gauge (Sir Wm. Thomson's Pattern).

The instrument consists of an astronomical clock, float-wheel and gear work for reducing the scale, and three drums, the whole fitted on a suitable plate and supporting standards, and requiring no further fixing. The clock is fitted with a compensated pendulum, and serves to show the time and to drive the centre or main drum of the instrument. The float-wheel is provided with pins which guide the copper band of the float as it coils itself during the rising tide. The right hand drum receives a reel of paper, and the paper is fitted to the instrument without further fixing. The left-hand or haul-off drum receives the paper records after it has passed round the main drum. The paper may be left to accumulate almost without limit on the haul-off drum, or can be removed at any time. The datum line on the record paper is traced by a fixed pencil, which can be adjusted to any level.

The main drum is furnished with a series of pins in distinctive positions, which perforate the paper as it passes over the drum, and thus give an absolute record of the time. The pencil-carrier is made to counterbalance the float-band when the scale is not too greatly reduced, in which case the weight of the float-band is partially relieved by a counterpoise weight acting on the axis of the float-wheel. The system of making the recording pencil balance the float-band is a great advantage, and accuracy of recording is secured.

The employment of a continuous roll of paper obviates the necessity of applying fresh paper to the recording drum, and the Tide Gauge can thus be left untended, except for the purpose of winding the clock, for an indefinite period.

The usual height of the drum is 18 inches, and, therefore, with a range of 16 feet, the scale is 1 inch per 1 foot.

The paper passes over the drum at the rate of 1 inch per hour, and a roll of paper lasts about 1 year.

INDEX

TO

THE SECOND PART.

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